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Drawing Shortest Paths in Geodetic Graphs[*]

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Abstract. Motivated by the fact that in a space where shortest paths are unique, no two shortest paths meet twice, we study a question posed by Greg Bodwin: Given a geodetic graph G, i.e., an unweighted graph in which the shortest path between any pair of vertices is a unique, is there a *philogeodetic* drawing of G, i.e., a drawing of G in which the curves of any two shortest paths meet at most once? We answer this question in the negative by showing the existence of geodetic graphs that require some pair of shortest paths to cross at least four times. The bound on the number of crossings is tight for the class of graphs we construct. Furthermore, we exhibit geodetic graphs of diameter two that do not admit a philogeodetic drawing.

Keywords: Edge crossings · Unique Shortest Paths · Geodetic graphs.

1 Introduction

Greg Bodwin [1] examined the structure of shortest paths in graphs with edge weights that guarantee that the shortest path between any pair of vertices is unique. Motivated by the fact that a set of unique shortest paths is *consistent* in the sense that no two such paths can "intersect, split apart, and then intersect again" he conjectured that if the shortest path between any pair of vertices in a graph is unique then the graph can be drawn so that any two shortest paths meet at most once. Formally, a *meet* of two curves $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$ is a maximal interval $I \subseteq [0,1]$ so that $\gamma_1(x) = \gamma_2(x)$, for all $x \in I$. A *crossing* is a meet with $I \cap \{0,1\} = \emptyset$. Two curves *meet k times* if they have k pairwise distinct meets.

[*] This research began at the Graph and Network Visualization Workshop 2019 (GNV'19) in Heiligkreuztal.

[**] Funded by the German Research Foundation DFG – Project-ID 50974019 – TRR 161 (B06).

[***] Supported by the Swiss National Science Foundation within the collaborative DACH project *Arrangements and Drawings* as SNSF Project 200021E-171681.

Shortest paths in simple polygons (geodesic paths) have the property that they meet at most once [6].

A *drawing* of graph G in $\mathbb{R}^2$ maps the vertices to pairwise distinct points and maps each edge to a Jordan arc between the two end-vertices that is disjoint from any other vertex. Drawings extend in a natural fashion to paths: Let φ be a drawing of G, and let $P = v_1, \ldots, v_n$ be a path in G. Then let $\varphi(P)$ denote the Jordan arc that is obtained as the composition of the curves $\varphi(v_1v_2), \ldots, \varphi(v_{n-1}v_n)$. A drawing φ of a graph G is *philogeodetic* if for every pair P_1, P_2 of shortest paths in G the curves $\varphi(P_1)$ and $\varphi(P_2)$ meet at most once.

An unweighted graph is *geodetic* if there is a unique shortest path between every pair of vertices. Trivial examples of geodetic graphs are trees and complete graphs. Observe that any two shortest paths in a geodetic graph are either disjoint or they intersect in a path. Thus, a planar drawing of a planar geodetic graph is philogeodetic. Any straight-line drawing of a complete graph is philogeodetic. It is a natural question to ask whether (geodetic) graphs can be drawn in such a way that the drawn shortest paths meet at most once.

Results. We show that there exist geodetic graphs that require some pair of shortest paths to meet at least four times (Theorem 1). This is even true in any topological drawing. The idea is to start with a sufficiently large complete graph and subdivide every edge exactly twice. The crossing lemma can be used to show that some pair of shortest paths must cross at least four times. By increasing the number of subdivisions per edge, we can reduce the density and obtain sparse counterexamples. The bound on the number of crossings is tight because any uniformly subdivided K_n can be drawn so that every pair of shortest paths meets at most four times (Theorem 2).

On one hand, our construction yields counterexamples of diameter five. On the other hand, the unique graph of diameter one is the complete graph, which is geodetic and admits a philogeodetic drawing (e.g., any straight-line drawing since all unique shortest paths are single edges). Hence, it is natural to ask what is the largest d so that every geodetic graph of diameter d admits a philogeodetic drawing. We show that $d = 1$ by exhibiting an infinite family of geodetic graphs of diameter two which do not admit philogeodetic drawings (Theorem 3). The construction is based on incidence graphs of finite affine planes. The proof also relies on the crossing lemma.

Related work. Geodetic graphs were introduced by Ore who asked for a characterization as Problem 3 in Chapter 6 of his book "Theory of Graphs" [7, p. 104]. An asterisk flags this problem as a research question, which seems justified, as more than sixty years later a full characterization is still elusive.

Stemple and Watkins [14,15] and Plesník [10] resolved the planar case by showing that a connected planar graph is geodetic if and only if every block is (1) a single edge, (2) an odd cycle, or (3) it stems from a K_4 by iteratively choosing a vertex v of the K_4 and subdividing the edges incident to v uniformly. Geodetic graphs of diameter two were fully characterized by Scapellato [12].

They include the Moore graphs [3] and graphs constructed from a generalization of affine planes. Further constructions for geodetic graphs were given by Plesník [10,11], Parthasarathy and Srinvasan [9], and Frasser and Vostrov [2].

Plesník [10] and Stemple [13] proved that a geodetic graph is homeomorphic to a complete graph if and only if it is obtained from a complete graph K_n by iteratively choosing a vertex v of the K_n and subdividing the edges incident to v uniformly. A graph is geodetic if it is obtained from any geodetic graph by uniformly subdividing each edge an even number of times [9,11].

2 Subdivision of a Complete Graph

The complete graph K_n is geodetic and rather dense. However, all shortest paths are very short, as they comprise a single edge only. So despite the large number of edge crossings in any drawing, every pair of shortest paths meets at most once, as witnessed, for instance, by any straight-line drawing of K_n. In order to lengthen the shortest paths it is natural to consider subdivisions of K_n.

As a first attempt, one may want to "take out" some edge uv by subdividing it many times. However, Stemple [13] has shown that in a geodetic graph every path where all internal vertices have degree two must be a shortest path. Thus, it is impossible to take out an edge using subdivisions. So we use a different approach instead, where all edges are subdivided uniformly.

Theorem 1. *There exists an infinite family of sparse geodetic graphs for which in any drawing in $\mathbb{R}^2$ some pair of shortest paths meets at least four times.*

Proof. Take a complete graph K_s, for some $s \in \mathbb{N}$, and subdivide every edge uniformly t times, for t even. The resulting graph $K(s,t)$ is geodetic. Note that $K(s,t)$ has $n = s + t\binom{s}{2}$ vertices and $m = (t+1)\binom{s}{2}$ edges, where $m \in O(n)$, for s constant and t sufficiently large. Consider an arbitrary drawing Γ of $K(s,t)$.

Let B denote the set of s branch vertices (that is, vertices of degree > 2) in $K(s,t)$. For two distinct vertices $u, v \in B$, let $[uv]$ denote the shortest uv-path in $K(s,t)$, which corresponds to the subdivided edge uv of the underlying K_s. As t is even, the path $[uv]$ consists of $t+1$, an odd number of edges. For every such path $[uv]$, with $u, v \in B$, we charge the crossings in Γ along the $t+1$ edges of $[uv]$ to one or both of u and v as detailed below; see Fig. 1 for illustration.

- Crossings along an edge that is closer to u than to v are charged to u;
- crossings along an edge that is closer to v than to u are charged to v; *and*
- crossings along the single central edge of $[uv]$ are charged to both u and v.

Let Γ_s be the drawing of K_s induced by Γ: every vertex of K_s is placed at the position of the corresponding branching vertex of $K(s,t)$ in Γ and every edge of K_s is drawn as a Jordan arc along the corresponding path of $K(s,t)$ in Γ. Assuming $\binom{s}{2} \geq 4s$ (i.e., $s \geq 9$), by the Crossing Lemma [8], at least

$$\frac{1}{64}\frac{\binom{s}{2}^3}{s^2} = \frac{1}{512}s(s-1)^3 \geq c \cdot s^4$$

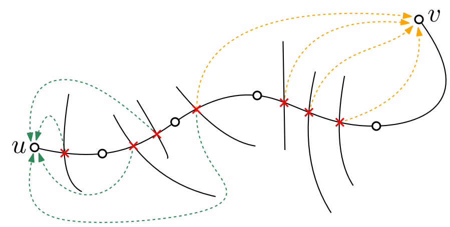

Fig. 1. Every crossing is charged to at least one endpoint of each of the two involved (independent) edges. Vertices are shown as white disks, crossings as red crosses, and charges by dotted arrows. The figure shows an edge uv that is subdivided four times, splitting it into a path with five segments. A crossing along any such segment is assigned to the closest of u or v. For the central segment, both u and v are at the same distance, and any crossing there is assigned to both u and v.

pairs of independent edges cross in Γ_s, for some constant c. Every crossing in Γ_s corresponds to a crossing in Γ and is charged to at least two (and up to four) vertices of B. Thus, the overall charge is at least $2cs^4$, and at least one vertex $u \in B$ gets at least the average charge of $2cs^3$.

Each charge unit corresponds to a crossing of two independent edges in Γ_s, which is also charged to at least one other vertex of B. Hence, there is a vertex $v \neq u$ so that at least $2cs^2$ crossings are charged to both u and v. Note that there are only $s-1$ edges incident to each of u and v, and the common edge uv is not involved in any of the charged crossings (as adjacent rather than independent edge). Let E_x, for $x \in B$, denote the set of edges of K_s that are incident to x.

We claim that there are two pairs of mutually crossing edges incident to u and v, respectively; that is, there are sets $C_u \subset E_u \setminus \{uv\}$ and $C_v \subset E_v \setminus \{uv\}$ with $|C_u| = |C_v| = 2$ so that e_1 crosses e_2, for all $e_1 \in C_u$ and $e_2 \in C_v$.

Before proving this claim, we argue that establishing it completes the proof of the theorem. By our charging scheme, every crossing $e_1 \cap e_2$ happens at an edge of the path $[e_1]$ in Γ that is at least as close to u as to the other endpoint of e_1. Denote the three vertices that span the edges of C_u by u, x, y. Consider the two subdivision vertices x' along $[ux]$ and y' along $[uy]$ that form the endpoint of the middle edge closer to x and y, respectively, than to u; see Fig. 2 for illustration.

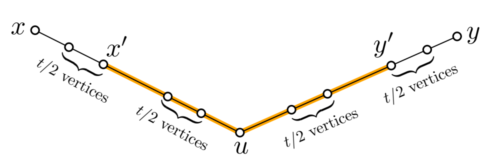

Fig. 2. Two adjacent edges ux and uy, both subdivided t times, and the shortest path between the "far" endpoints x' and y' of the central segments of $[ux]$ and $[uy]$.

The triangle uxy in K_s corresponds to an odd cycle of length $3(t+1)$ in $K(s,t)$. So the shortest path between x' and y' in $K(s,t)$ has length $2(1+t/2) = t+2$ and passes through u, whereas the path from x' via x and y to y' has length $3(t+1)-(t+2) = 2t+1$, which is strictly larger than $t+2$ for $t \geq 2$. It follows that the shortest path between x' and y' in $K(s,t)$ is crossed by both edges in C_v. A symmetric argument yields two branch vertices a' and b' along the two edges in C_v so that the shortest $a'b'$-path in $K(s,t)$ is crossed by both edges in C_u. By definition of our charging scheme (that charges only "nearby" crossings to a vertex), the shortest paths $x'y'$ and $a'b'$ in $K(s,t)$ have at least four crossings.

It remains to prove the claim. To this end, consider the bipartite graph X on the vertex set $E_u \cup E_v$ where two vertices are connected if the corresponding edges are independent and cross in Γ_s. Observe that two sets C_u and C_v of mutually crossing pairs of edges (as in the claim) correspond to a 4-cycle C_4 in X. So suppose for the sake of a contradiction that X does not contain C_4 as a subgraph. Then by the Kővári-Sós-Turán Theorem [5] the graph X has $O(s^{3/2})$ edges. But we already know that X has at least $2cs^2 = \Omega(s^2)$ edges, which yields a contradiction. Hence, X is not C_4-free and the claim holds. □

The bound on the number of crossings in Theorem 1 is tight.

Theorem 2. *Any uniformly subdivided (an even number of times) K_n can be drawn so that every pair of shortest paths crosses at most four times.*

We only sketch the construction, a proof of Theorem 2 can be found in the appendix. Place the vertices in convex position, and draw the subdivided edges along straight-line segments. For each edge, put half of the subdivision vertices very close to one endpoint and the other half very close to the other endpoint (Fig. 3). As a result, all crossings fall into the central segment of the path.

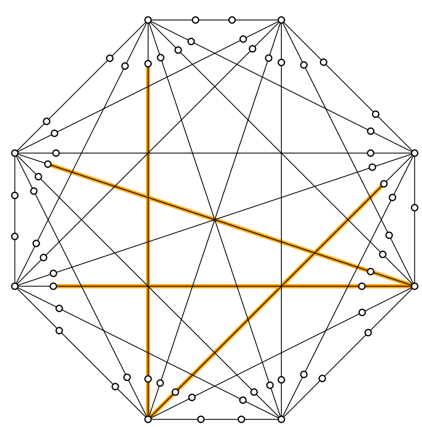

Fig. 3. A drawing of $K(8,2)$, the complete graph K_8 where every edge is subdivided twice, so that every pair of shortest paths meets at most four times. Two shortest paths that meet four times are shown bold and orange.

3 Graphs of Diameter Two

In this section we give examples of geodetic graphs of diameter two that cannot be drawn in the plane such that any two shortest paths meet at most once.

An *affine plane* of order $k \geq 2$ consists of a set of lines and a set of points with a containment relationship such that (i) each line contains k points, (ii) for any two points there is a unique line containing both, (iii) there are three points that are not contained in the same line, and (iv) for any line ℓ and any point p not on ℓ there is a line ℓ' that contains p, but no point from ℓ. Two lines that do not contain a common point are *parallel*. Observe that each point is contained in $k+1$ lines. Moreover, there are k^2 points and $k+1$ classes of parallel lines each containing k lines. The 2-dimensional vector space $\mathbb{F}^2$ over a finite field $\mathbb{F}$ of order k with the lines $\{(x, mx+b);\ x \in \mathbb{F}\}$, $m, b \in \mathbb{F}$ and $\{(x_0, y);\ y \in \mathbb{F}\}$, $x_0 \in \mathbb{F}$ is a finite affine plane of order k. Thus, there exists a finite affine plane of order k for any k that is a prime power (see, e.g., [4]).

Scapellato [12] showed how to construct geodetic graphs of diameter two as follows: Take a finite affine plane of order k. Let L be the set of lines and let P be the set of points of the affine plane. Consider now the graph G_k with vertex set $L \cup P$ and the following two types of edges: There is an edge between two lines if and only if they are parallel. There is an edge between a point and a line if and only if the point lies on the line; see Fig. 4. There are no edges between points. In Appendix B, we prove that G_k is a geodetic graph of diameter two.

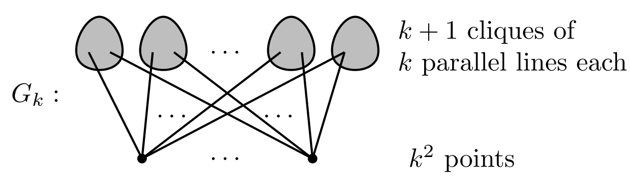

Fig. 4. Structure of the graph G_k.

Theorem 3. *There are geodetic graphs of diameter two that cannot be drawn in the plane such that any two shortest paths meet at most once.*

Proof. Let $k \geq 129$ be such that there exists an affine plane of order k (e.g., the prime $k = 131$). Assume there was a drawing of G_k in which any two shortest paths meet at most once. Let G be the bipartite subgraph of G_k without edges between lines. Observe that any path of length two in G is a shortest path in G_k. As G has $n = 2k^2 + k$ vertices and $m = k^2(k+1) > kn/2$ edges, we have $m > 4n$, for $k \geq 8$. Therefore, by the Crossing Lemma [8, Remark 2 on p. 238] there are at least $m^3/64n^2 > k^3n/512$ crossings between independent edges in G.

Hence, there is a vertex v such that the edges incident to v are crossed more than $k^3/128$ times by edges not incident to v. By assumption, (a) any two edges meet at most once, (b) any edge meets any pair of adjacent edges at most once, and (c) any pair of adjacent edges meets any pair of adjacent edges at most once. Thus, the crossings with the edges incident to v stem from a matching. It follows that there are at most $(n-1)/2 = (2k^2+k-1)/2$ such crossings. However, $(2k^2+k-1)/2 < k^3/128$, for $k \geq 129$. □

4 Open Problems

We conclude with two open problems: (1) Are there diameter-2 geodetic graphs with edge density $1+\varepsilon$ that do not admit a philogeodetic drawing? (2) What is the complexity of deciding if a geodetic graph admits a philogeodetic drawing?

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A Proof of Theorem 2

Proof. Draw the graph as described on Page 5 and as illustrated in Fig. 3 for $K(8,2)$. There are two different types of vertices, and six different types of shortest paths. Let B denote the set of branch vertices, and let S denote the set of subdivision vertices. Note that for every edge uv of K_n, only the central segment of the subdivided path $[uv]$ may have crossings in the drawing. We claim that every shortest path in the graph contains at most two central segments in the drawing, from which the theorem follows immediately. Consider a pair u, v of vertices.

Case 1: $\{u,v\} \cap B \neq \emptyset$. Suppose without loss of generality that $u \in B$. If $v \in B$ or $v \in S$ subdivides an edge incident to u, then the shortest uv-path contains at most one central segment. Otherwise, $v \in S$ subdivides an edge xy disjoint from u. One of x or y, without loss of generality x is closer to v. Then the shortest uv-path is $[vx][xu]$, which contains exactly one central segment, $[xu]$.

Case 2: $u, v \in S$. If u and v subdivide the same edge, then the shortest uv-path contains at most one central segment. If u and v subdivide distinct adjacent segments, xy and xz, then the shortest uv-path is either $[ux][xv]$, which contains at most two central segments. Or the sum of the length of $[uy]$ and $[zv]$ is at most half of the number of subdivision vertices per edge and the shortest uv-path is $[uy][yz][zv]$, which then contains at most one central segment. Otherwise, u and v subdivide disjoint segments, xy and wz, where without loss of generality x is closer to u than y and w is closer to v than z. Then the shortest uv-path is $[ux][xw][wv]$, which contains exactly one central segment, $[xw]$. □

B Proof that G_k (as Defined in Section 3) is Geodetic

Lemma 1. *G_k is a geodetic graph of diameter two.*

Proof. Two lines have distance one if they are parallel. Otherwise they share exactly one vertex and, hence, are connected by exactly one path of length two. For any two points there is exactly one line that contains both. Given a line ℓ and a point p then either p lies on ℓ and, thus, p and ℓ have distance one. Or there is exactly one line ℓ' containing p that is parallel to ℓ and, thus, there is exactly one path of length two between ℓ and p. □