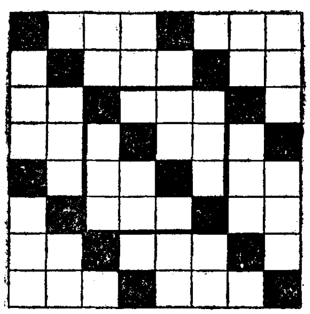
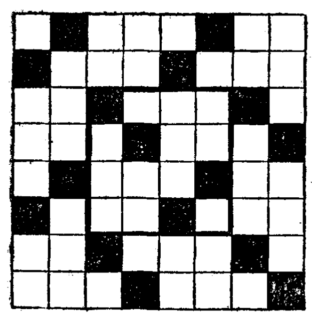
ON THE NUMBERS OF PATTERNS WHICH CAN BE DERIVED FROM CERTAIN ELEMENTS.

By *Professor J. E. A. Steggall.*

A SQUARE being divided into n^2 equal squares, and n of these being shaded and placed in such a way that no two shall be in the same row or column, we get what may be called an *element* of a pattern. If similar elements be, in turn, repeated by the addition lengthwise and breadthwise of other squares identical with the first, we get a *pattern*. It is clear that the number of *elements* is $n!$; the number of *patterns* derivable from the elements is much fewer. For example, taking n equal to 4, we have only three possible patterns, which may be expressed thus:—(1234), (1432), (1243), of which the first and second are left- and right-handed; as shown in figure



(1234)
The image of this gives (1432)



(1243)

Let $a_1a_2...a_n$ be any element, and $b_1b_2...b_n$ any other element, the a's and b's being simply the first n natural numbers in any orders. It is clear that these give the same pattern (1) if both elements are the same cyclical permutation of the numbers, *e.g.* (134625) and (462513) give identical patterns; (2) if by adding any constant k to the a's we obtain the b's in the same order, *e.g.* (134625) and (356241) give identical patterns, n being in this case 6, and k being 2. Thus a pattern can be represented in n ways by a horizontal change (mere cyclical transformation); but we have still to find in how many ways by a vertical change (addition of a constant number) the elements of the same pattern can be transformed non-cyclically.

Let the arrangement $a_1a_2...a_n$ be transformed to $a_{r+1}a_{r+2}...a_{n+r}$, where of course $a_{n+r}=a_r$, in the same cyclical order, by the addition of k to each term. Then if k is not a factor of n it is clear that the same arrangement, as regards cyclical order, will be arrived at by adding a smaller number than k: *i.e.* the greatest common factor of k and n. Hence we may now consider k to be a factor of n, say $n=pk$.

We therefore get the series of equations

$$a_{r+1}=a_1+k, \quad a_{r+2}=a_2+k, \quad ..., \quad a_{r+n}=a_n+k,$$
$$a_{2r+1}=a_1+2k, \quad a_{2r+2}=a_2+2k, \quad ..., \quad a_{2r+n}=a_n+2k,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{(p-1)r+1}=a_1+(p-1)k, \quad ..., \quad a_{(p-1)r+n}=a_n+(p-1)k,$$
$$a_{pr+1}=a_1+pk, \quad ..., \quad a_{pr+n}=a_n+pk;$$

the cycle being now completed.

From this table we infer

(1) That pr is a multiple of n, and therefore r of k;

(2) That $r, 2r, 3r, ... (p-1)r$ are all incongruent (mod n), for otherwise we should have, say, $(t-1)k$ divisible by n; hence r must equal qk, where q is prime to p;

(3) The groups beginning with $a_1, a_{r+1}, a_{2r+1}, ..., a_{(p-1)r+1}$, respectively, are identical, in some order, with the groups beginning

$$a_{k+1}, \; a_{2k+1}, \; a_{3k+1}, \; ..., \; a_{(p-1)k+1};$$

(4) The array

$$\begin{array}{llll} a_1, & a_2, & ..., & a_k, \\ a_1+k, & a_2+k, & ..., & a_k+k, \\ a_1+2k, & a_2+2k, & ..., & a_k+2k, \\ ... & ... & ... & ... \\ a_1+(p-1)k, & a_2+(p-1)k, & ..., & a_k+(p-1)k, \end{array}$$

includes the whole series of nnmbers

$$a_1, \; a_2, \; ..., \; a_n.$$

Hence $a_1, a_2, ..., a_k$ are all incongruent (mod k), and may therefore be taken as equal to the natural numbers 1 to k in any order, each having any multiple of k added to it. In other words the terms $a_1, a_2, ..., a_k$ may be taken as equal to $b_1+m_1k, b_2+m_2k, ..., b_k+m_kk$, where $b_1, b_2, ..., b_k$ are, in any order, the numbers 1 to k; and $m_1, m_2, ..., m_k$ are any integers, repeated or not, from 0 to $p-1$ inclusive.

Now every value of r, or of q, gives a different sequence to the groups of k; and thus the number of arrangements of the numbers 1 to n such that the addition of k to each leaves the cyclical order unchanged is $U_k = \phi(p) \times p^k \times k!$, where $\phi(p)$ denotes the number of numbers less than, and prime to, p.

This number of arrangements clearly includes all those that repeat not only when k is added, but also all those that repeat when any submultiple of k is added; to discriminate between the two we have to solve some simultaneous equations.

If k is the least number that when added to the constituents of any arrangement causes it to repeat cyclically, the pattern can be represented in nk ways. Hence the total number of possible patterns is equal to the sum of all the terms

$$\frac{\text{number of arrangements repeating by addition of } k \text{ and not before}}{nk}.$$

Now if U_k = number of arrangements that repeat after addition of k, and submultiples, u_k = number of arrangements that repeat after addition of k only, we have

$$U_k = u_1 + u_a + u_b + u_{ab} + \ldots + u_k,$$

where each subscript is a factor of k.

For example

$$\begin{aligned} U_{12} &= u_1 + u_2 + u_3 + u_4 + u_6 + u_{12}, \\ U_6 &= u_1 + u_2 + u_3 + u_6, \\ U_4 &= u_1 + u_2 + u_4, \\ U_3 &= u_1 + u_3, \\ U_2 &= u_1 + u_2, \\ U_1 &= u_1, \end{aligned}$$

giving six equations to determine the u's in terms of the U's, which latter are readily found by the formulæ.

The solutions here are

$$\begin{aligned} u_1 &= U_1, \\ u_2 &= U_2 - U_1, \\ u_3 &= U_3 - U_1, \\ u_4 &= U_4 - U_2, \\ u_6 &= U_6 - U_3 - U_2 + U_1, \\ u_{12} &= U_{12} - U_6 - U_4 + U_2. \end{aligned}$$

A few special cases may be examined:—

(1) n prime, $k=1$, $p=n$, or $k=n$, $p=1$.

$$U_n = u_n + u_1 = n!,$$
$$U_1 = \quad u_1 = n(n-1);$$

whence
$$u_n = n! - n(n-1).$$

Therefore number of patterns

$$= \frac{n! - n(n-1)}{n^2} + \frac{n(n-1)}{n}$$
$$= \frac{(n-1)! + (n-1)^2}{n}.$$

This gives for $n = 2, 3, 5, 7$, respectively, 1, 2, 8, 108.

(2) $n = ab$, a and b prime.

Here

$$U_{ab} = u_1 + u_a + u_b + u_{ab} = n!,$$
$$U_a = u_1 + u_a = a!\, b^a (b-1),$$
$$U_b = u_1 + u_b = b!\, a^b (a-1),$$
$$U_1 = u_1 = ab(a-1)(b-1);$$

therefore

$$u_a = a(a-1)\, b(b-1)\{(a-2)!\, b^{a-1} - 1\},$$
$$u_b = b(b-1)\, a(a-1)\{(b-2)!\, a^{b-1} - 1\},$$
$$u_{ab} = (ab)! - ab(a-1)(b-1)\{(a-2)!\, b^{a-1} + (b-2)!\, a^{b-1} - 1\},$$

e.g. $n = 6$, $a = 2$, $b = 3$;

$$u_1 = 12, \quad u_2 = 24, \quad u_3 = 36, \quad u_6 = 648.$$

Number of patterns $\frac{648}{36} + \frac{36}{18} + \frac{24}{12} + \frac{12}{2} = 24.$

(3) $n = a^2$, a prime,

$$U_n = u_1 + u_a + u_n = n!,$$
$$U_a = u_1 + u_a = a!\,(a-1)\, a^a,$$
$$U_1 = u_1 = a^3(a-1);$$

therefore
$$u_a = a^3(a-1)\{(a-1)!\, a^{a-2} - 1\},$$
$$u_n = n! - a!\, a^a (a-1),$$

e.g.
$$a = 2, \; n = 4,$$
$$u_1 \qquad = 8,$$
$$u_1 + u_2 \qquad = 8,$$
$$u_1 + u_2 + u_4 = 24;$$

therefore $u_4 = 16, \quad u_2 = 0, \quad u_1 = 8.$

Therefore number of patterns $= \frac{16}{16} + \frac{0}{8} + \frac{8}{4} = 4.$

(4) $n = 2^r,$

$$U_n = u_1 + u_2 + u_4 \ldots + u_{2^r} = n!,$$
$$U_{\frac{1}{2}n} = u_1 + u_2 + \ldots + u_{2^{r-1}} = 2^{r-1}!\, 2^{2^{r-1}},$$
$$U_{\frac{1}{4}n} = u_1 + u_2 + \ldots + u_{2^{r-2}} = 2^{r-2}!\, .\, 4^{2^{r-2}}.\, 2,$$
$$U_1 = u_1 = 2^r.\, 2^{r-1}.$$

Hence

$$u_n = U_n - U_{\frac{1}{2}n},$$
$$u_{\frac{1}{2}n} = U_{\frac{1}{2}n} - U_{\frac{1}{4}n},$$
$$\ldots = \ldots$$
$$u_2 = U_2 - U_1,$$
$$u_1 = U_1 .$$

Therefore number of patterns

$$= \frac{u_n}{n^2} + \frac{2u_{\frac{1}{2}n}}{n^2} + \frac{4u_{\frac{1}{4}n}}{n^2} + \ldots + \frac{2^r U_1}{n^2}$$
$$= \frac{U_n + U_{\frac{1}{2}n} + 2U_{\frac{1}{4}n} + \ldots + 2^{r-1} U_1}{n^2},$$

e.g. $r = 3, \; n = 8,$

$$u_1 + u_2 + u_4 + u_8 = 8!,$$
$$u_1 + u_2 + u_4 = 4!\, .\, 2^4,$$
$$u_1 + u_2 = 2!\, .\, 4^2.\, 2,$$
$$u_1 = 8.4,$$
$$u_2 = 32,$$
$$u_4 = 320,$$
$$u_8 = 2^4.\, 104.\, 4!.$$

Therefore number of patterns

$$= \frac{2^4.\, 104.\, 4!}{64} + \frac{320}{32} + \frac{32}{16} + \frac{32}{8}$$
$$= 624 + 10 + 2 + 4 = 640.$$

The following table shows the number of arrangements that repeat, for the first time, on addition of k to each constituent for all values of n from 1 to 12.

n	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	Patterns
1	1	.											1
2	2	.											1
3	6	.	0										2
4	8	0	.	16									3
5	20	.	.	.	100								8
6	12	24	36	.	.	648							24
7	42	.	.	.	.	.	4998						108
8	32	32	.	320	.	.	.	39936					640
9	54	.	270	.	.	.	.	.	362556				4492
10	40	160	.	.	3800	.	.	.	.	3624800			36336
11	110	.	.	.	.	.	.	.	.	.	39916990		329900
12	48	96	720	3744	.	45216	.	.	.	.	.	478951776	3326788