THE MATHEMATICAL THEORY OF THE SATEEN ARRANGEMENT.

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Geometrically considered the sateen arrangement is a certain regular method of arranging points on squared paper, each point being placed at the centre of a square. In textile work the sateen arrangement is used, (1) as a basis for actual weaves, (2) as a mode of distributing over the surface of a cloth the repeats of an ornamental design.

We may regard the sateen arrangement as being generated in the following manner. Let us place on squared paper a point A in a certain square * (see Fig. 1), then a point B in the next column of squares, a certain number of, say two, squares higher, a point C in the next column the same number of squares higher than B, and so on. We thus obtain what is known as a "twill" or "diagonal." Now suppose that by a thickening of some of the lines of the squared paper, the paper is divided into larger squares whose sides are, say, five times as long as those of the small squares. Now suppose that the arrangement is one which repeats itself in each of the large squares. Placing D' and E' in the large square containing A to correspond with D and E, we obtain one complete element of the pattern which may be repeated *ad lib.* over the rest of the paper.

In the above case I have chosen 5 as the number of times the side of the small square is contained in that of the large square, and 2 as the number of squares the "twill" ascends from one column to the other. The resulting design is a sateen of order 5 (or a 5-end sateen) with a step of 2 (or stepping 2). These two numbers, the "order" and the "step," completely specify a sateen. We may not, however, choose any two numbers. In order to obtain a weavable design we must put a point in every row and every column of the design square. Thus, if we make the twill of Fig. 1 repeat on a design square of 6, we obtain the result shown in Fig. 2, which does not fulfil this condition.

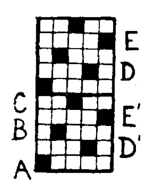

Fig. 1.

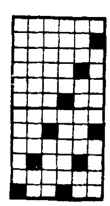

Fig. 2.

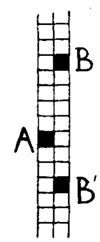

Fig. 3.

This is owing to the fact that the points begin to repeat on the rows of the design square before all the columns have been filled. In order that this may not happen the L.C.M. of the order and step must be equal to their product. Hence the first well-known sateen rule—the order and the step must be prime to each other.

There is a second well-known rule which relates to the lateral inversion of the design. Let A, B and B' be three points in a sateen arrangement (see Fig. 3), B and B' being in a column adjacent to A, and B' being an adjacent repeat of B so that the distance BB' is equal to the order of the sateen. Now we may suppose the sateen to have been generated (a) by stepping in the direction AB (5 squares upwards to the right), (b) by stepping in the direction

* In practice these "points" are expanded into solid black squares.

B′ (3 squares downwards to the right). If a step of this latter magnitude ere taken in the same direction as the former step, we should evidently obtain e mirror image of the sateen. Moreover, the sum of the steps is equal to e order of the sateen. Such steps are said to be complementary. We us see that complementary steps give sateens which are the mirror images each other. This is the second well-known sateen rule. In Fig. 4 is shown

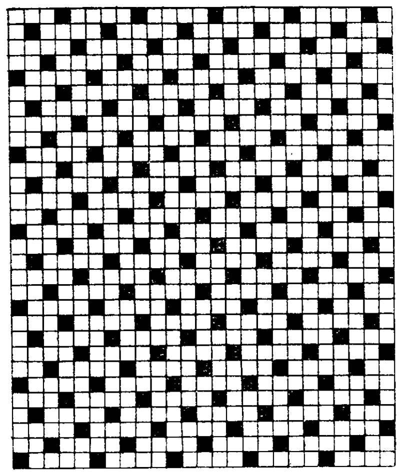

Fig. 4.

the 5/3 sateen,* which, in accordance with this rule, is the mirror image of the 5/2 sateen shown in Fig. 1.

These two rules seem to comprise the whole of the hitherto published systematic knowledge of the subject. There is in existence no general theory of sateens, in spite of the importance of the subject in textile designing. The object of the present paper is to formulate a mathematical theory of the subject and arrive at the general principles underlying those properties of sateens which are of importance in designing.

The first important question is that of the number of essentially distinct sateens of any given order. The first rule limits the possible values of the step to numbers prime to the order. The second rule shows that only half of these possible sateens are essentially distinct designs. There is, however, another limitation. It is evident that the rotation of a sateen through a right angle will in general give another sateen with a different step (and, of course, of the same order). The relation between the steps of two sateens derivable one from the other by a rotation through a right angle is readily investigated. In Fig. 5 the generating twill of a sateen is continued till we reach a point H in the top row of a design square, *i.e.* one row below a repeat (A'') of the initial point A. Now we may regard the sateen as being generated by stepping in the direction $A''H$. A rotation through a right angle converts this into the normal "upward to the right" mode of stepping. Now it is evident that the product of the new step and the old is less by unity than a multiple of the order. Hence the condition for two steps s_1 and s_2 to give sateens derivable one from the other by a right angle rotation is that

$$s_1 s_2 = in - 1,$$

where i is an integer.

* For the sake of brevity we will call the sateen of order n and step s the n/s sateen.

A consideration of the step $A'E$ (Fig. 5) will readily show that if the condition

$$s_1 s_2 = in + 1$$

is fulfilled, the rotation of one sateen gives the mirror image of the other.

We thus have a third rule limiting the number of essentially distinct patterns. An application of the three rules shows that even for high orders the number of patterns is in general small. For example, consider sateens of order 20. Applying the first rule we obtain the possible steps *

3, 7, 9, 11, 13, 17.

We may cut out the second three as yielding mirror images of the first three; and since $3 \times 7 = 20 + 1$, we may cut out the 7. We thus have only two distinct patterns given by the 3 and the 9 (or by the 7 and 11, 13 and 9, etc.).

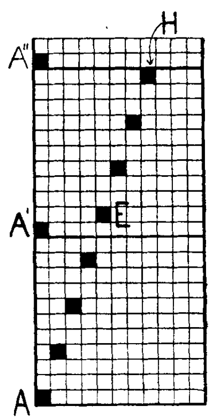

FIG. 5.

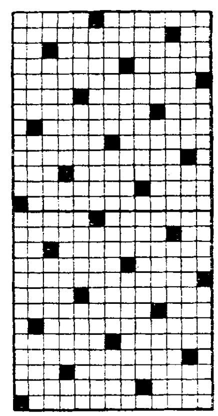

FIG. 6.

We obviously have the best chance of obtaining a large number of patterns when the order is a prime number. Thus in the case of the order 13, the steps

2, 3, 4, 5, 6,

left by the application of the second rule, may be reduced to

2, 3, 5

by the application of the third rule, so that even in this case there are only 3 distinct patterns. It will be seen that the 2 cuts out the 6 and the 3 the 4. The step 5 is peculiar; we have $5^2 = 2 \times 13 - 1$. Hence the 13/5 sateen is unaffected by a rotation through a right angle. This sateen is shown in Fig. 6. It will be noticed that the points lie at the intersections of a square lattice, which, as can be shown from elementary considerations, is not altered in appearance by rotation through a right angle.

We will not consider further the question of the number of sateens of any given order, but will proceed to the consideration of what may be termed the "sateen lattice." We may regard the points of a sateen arrangement as lying at the intersections of two series of equidistant parallel lines. This "sateen lattice" is not unique for any given sateen. Every sateen may be regarded as giving rise to an infinity of lattices, though the number actually suggested to the eye by any sateen is limited. Thus in the 17/10 sateen indicated in Fig. 7, the elementary parallelograms of two possible lattices are shown.

* The step of one, or one less than the order, gives a continuous twill which is not regarded as a sateen.

In distributing an ornamental figure on a sateen basis, the shape, size and orientation of such parallelograms are obviously of importance in relation to the shape, size and orientation of the figure. It may perhaps not be out of place to point out at this stage the characteristic virtue of the sateen distribution—not possessed by simpler arrangements (such as the "half-drop"

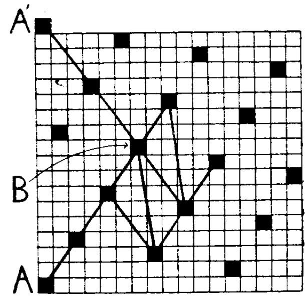

FIG. 7.

arrangement, much used in the design of wallpapers, which is founded on a lattice arrangement whose elementary equilateral parallelogram has its two diagonals respectively horizontal and vertical). A sateen arrangement which possesses a square or nearly square lattice does not obtrude its regularity on the eye, and may even produce a pleasing effect of irregularity.

Not every lattice arrangement gives rise to a sateen. If we suppose the lattice to be specified in shape and size of mesh and orientation by the four measurements a, b, c, d indicated in Fig. 8, it is necessary that a should be

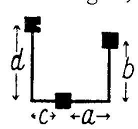

FIG. 8.

prime to c, and b to d. This follows from elementary considerations. We may thus generate a sateen from an elementary parallelogram specified by any four numbers, a, b, c, d fulfilling this condition. The following problem immediately suggests itself. What is the relation between the order n and step s on the one hand, and the four numbers a, b, c, d? A purely empirical way of solving the problem in a concrete case would be to fill in the design according to the specified plan until two points were obtained, (a) in the same column, (b) in adjacent columns. The vertical distance between the points would give in case (a) the order, and in case (b) the step (or its complement). A systematic method of obtaining the order is shown in Fig. 7. Let us start from a point A along an "upwards to the right" lattice line till we arrive at a point B whose horizontal distance from the starting-point, which is obviously a multiple of a, is also a multiple of c. Since a and c are prime to each other, this distance must be ac. If now we proceed back towards the initial column along an "upwards to the left" lattice line, we shall obtain a point A' vertically above A. The first part of the journey consists of c stages, each with a vertical component b, and the second part of a stages each with a vertical component d. Hence the total vertical component is $bc+ad$. Hence we have for the order,

$$n=ad+bc$$

or

$$n=\begin{vmatrix} a & b \\ -c & d \end{vmatrix},$$

which is a very simple relation between the order and the rectangular coordinates of the diametrically opposite corners of an elementary parallelogram relative to one of the remaining corners.

For the same sateen we may have an infinity of sets of values of a, b, c, d, which should give, according to the above formula, the same value of n. If we consider the series of parallelograms on the same base and between the same parallel lattice lines (two consecutive members of which are shown in Fig. 7), we see that the passage from one member to the next corresponds to the addition of members of one row of the determinant to the corresponding members of the other row, a process which does not alter the value of the determinant.

The value of the step may be determined in similar manner. If h stages upwards to the right and k stages upwards to the left terminate at a point in a column immediately to the right of that occupied by the initial point, we have

$$ha - kc = 1.$$

The step is given by

$$s = hb + kd.$$

Eliminating k, and making use of the formula for the order, we obtain the relation

$$s = \frac{hn - d}{c},$$

in which h is the least integer which will give an integral value for s.

In Fig. 7, for the "squarer" of the two lattices indicated, a, b, c and d have respectively the values 2, 3, 3 and 4. Our formula for the order gives the value 17. For the step we have

$$s = \frac{17h - 4}{3},$$

which for $h = 2$ gives $s = 10$. Hence the sateen is of order 17 and step 10.

We will conclude by considering one more question of the many arising out of a consideration of the sateen arrangement. This question relates to what may be termed the "twilliness" of a sateen. Certain sateens suggest to the eye a definite "twill," while others are devoid of any such suggestion. For most purposes for which the sateen arrangement is used in textile work this suggestion is a defect, though for certain special effects a marked "twilliness" may be desirable. A comparison of the 17/2 sateen shown in Fig. 9 with the 5/3 sateen shown in Fig. 4 will reveal the origin of "twilliness." If

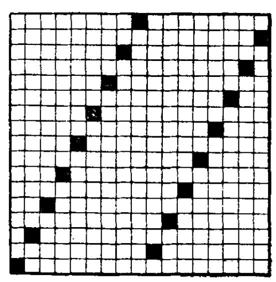

FIG. 9.

the most nearly rectangular parallelogram has very unequal pairs of sides, then there is a marked twill in the direction of the shorter sides, the nearness of the points suggesting to the eye a continuous line. If the sateen yields a square lattice, then we get a minimum of "twilliness." Such sateens are generally called "perfect" sateens, since they possess in the most marked degree that combination of the regular and the chaotic which was the primary object of the invention of the sateen arrangement.

The condition for a square sateen is evidently that d and c should be respectively equal to a and b. Hence we have

$$n = a^2 + b^2,$$

i.e. the order of a square sateen is equal to the sum of the squares of two numbers which are prime to each other. The step may be deduced from the relation

$$s^2 = in - 1,$$

which expresses the fact that the sateen is not altered by rotation through a right angle. The smallest value of i giving an integral value of s, gives the smaller of the two complementary steps.

The following table shows the order and step of the first seven "perfect" sateens:

Order.	Step.
$5(1^2+2^2)$	2
$10(1^2+3^2)$	3
$13(2^2+3^2)$	5
$17(1^2+4^2)$	4
$25(3^2+4^2)$	7
$26(1^2+5^2)$	5
$29(2^2+5^2)$	12

In the general case the ratio of the distances between consecutive members of the two sets of lattice lines is evidently

$$\sqrt{\frac{a^2+b^2}{c^2+d^2}}.$$

Hence the deviation from the value unity of this quantity in the case of the most nearly rectangular lattice given by any sateen affords a kind of numerical measure of the "twilliness." *

There are many other problems of a mathematical nature connected with the sateen arrangement, and with textile designing in general, but the present paper will, I hope, be sufficient to show that even so technical a subject as textile designing presents many problems of interest to the pure mathematician. S. A. S.

* Assuming that a square sateen has "twilliness" of zero, the most suitable measure of the "twilliness" would of course be $\log \sqrt{a^2+b^2}/\sqrt{c^2+d^2}$.

50. . . . During these graduate years at Pembroke Hall . . . Mr. Pitt laid in his principal stores of knowledge . . . In mathematics, the especial pride of Cambridge, he took great delight. He frequently alluded in later life to the practical advantage he had derived from them, and declared that no portion of his time had been more usefully employed than that which he devoted to this study. He was master of everything usually known by the academic "wranglers," and felt a great desire—but Mr. Pretyman did not think it right to indulge the inclination—to fathom still farther the depths of pure mathematics. "When," adds Mr. Pretyman, "the connection of tutor and pupil was about to cease between us, he expressed a hope that he should find leisure and opportunity to read Newton's *Principia* again with me after some summer circuit."—[George Pretyman, Sen. Wrangler, 1772, tutor of Pembroke Hall, tutor of William Pitt, in 1803 took name of Tomline. Bishop Winchester, 1820 ; *d.* 1827, *b.* 1750, Bury St. Edmunds.]