Characterization of Unlabeled Level Planar Graphs

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The 15th International Symposium on Graph Drawing (GD 2007)

Outline

- Background

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  - Motivation
  - Definitions
  - Previous Work
  - New Results

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- Unlabeled Level Planar Trees

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  - Forbidden Trees

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  - Drawing Algorithms

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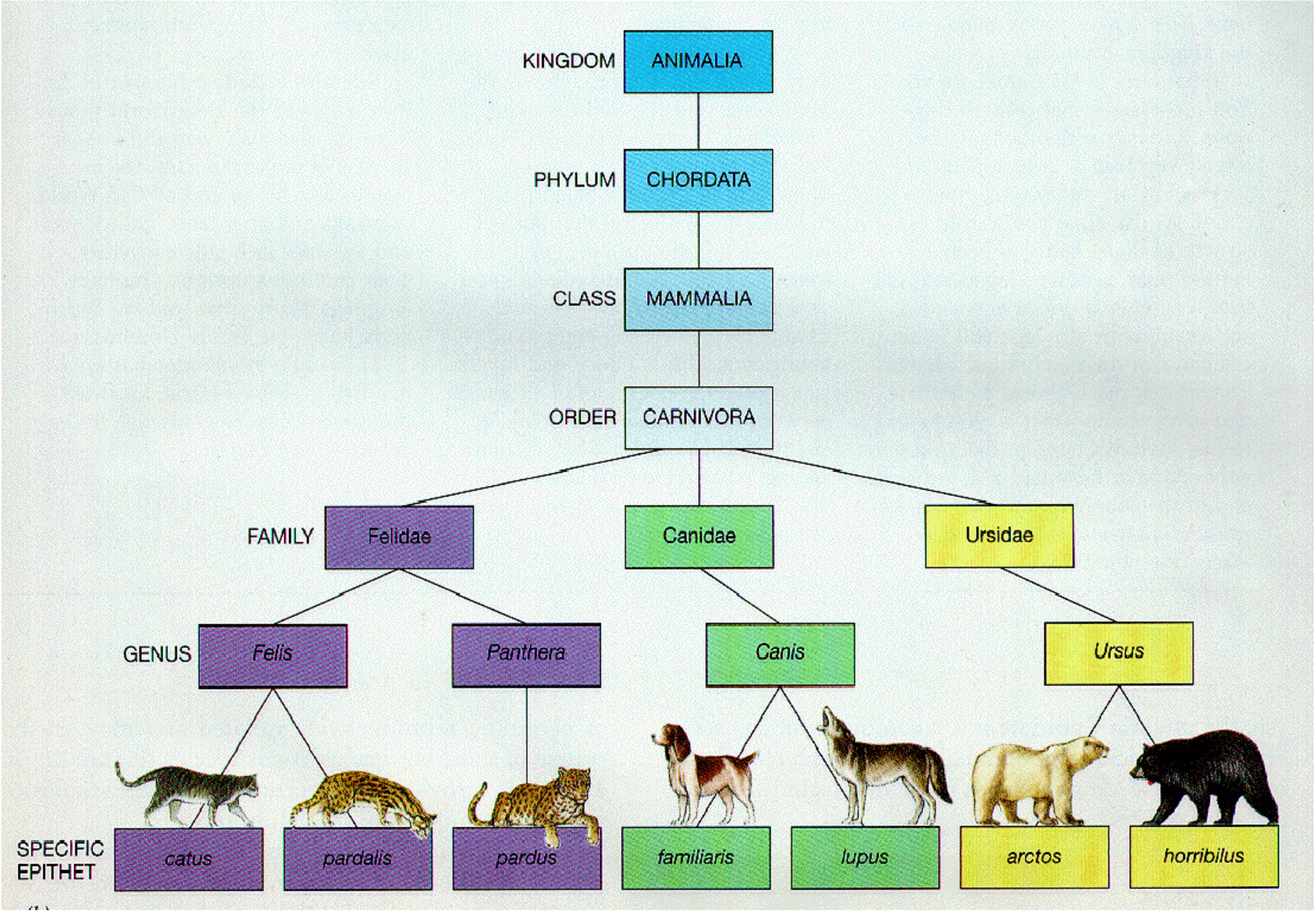
    - Classic animal taxonomy



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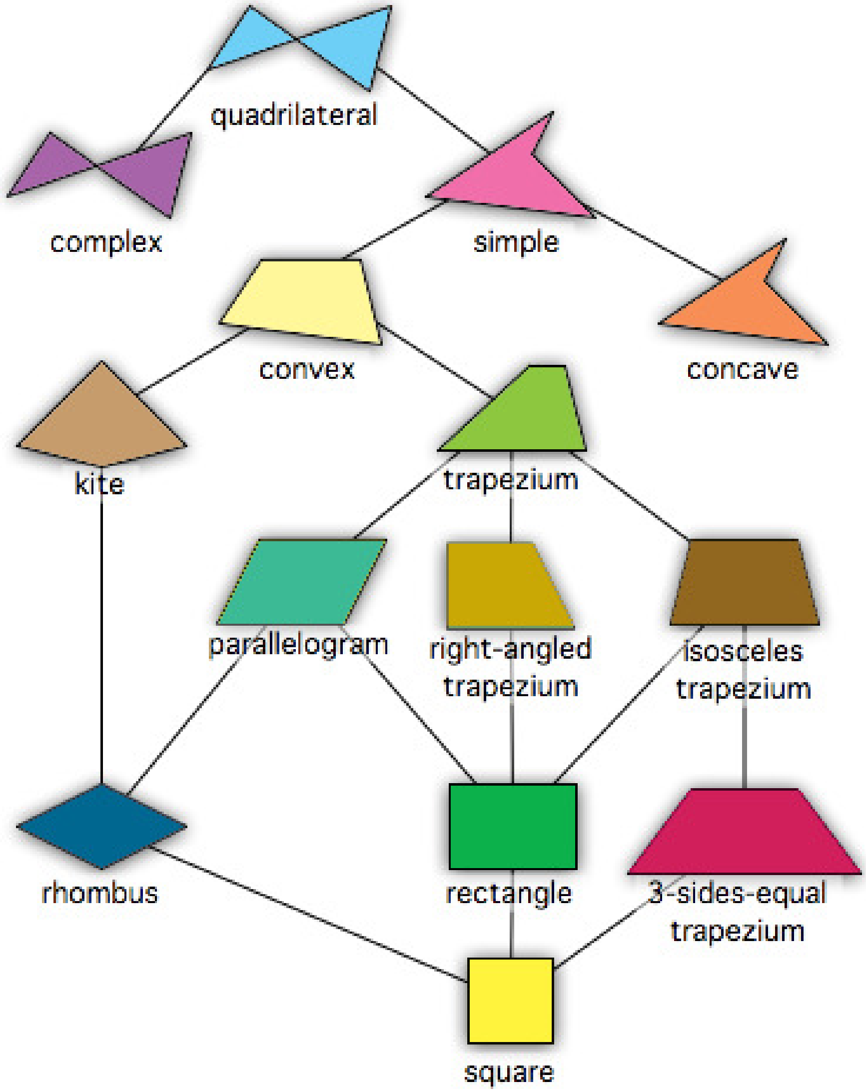
    - Polygon hierarchy



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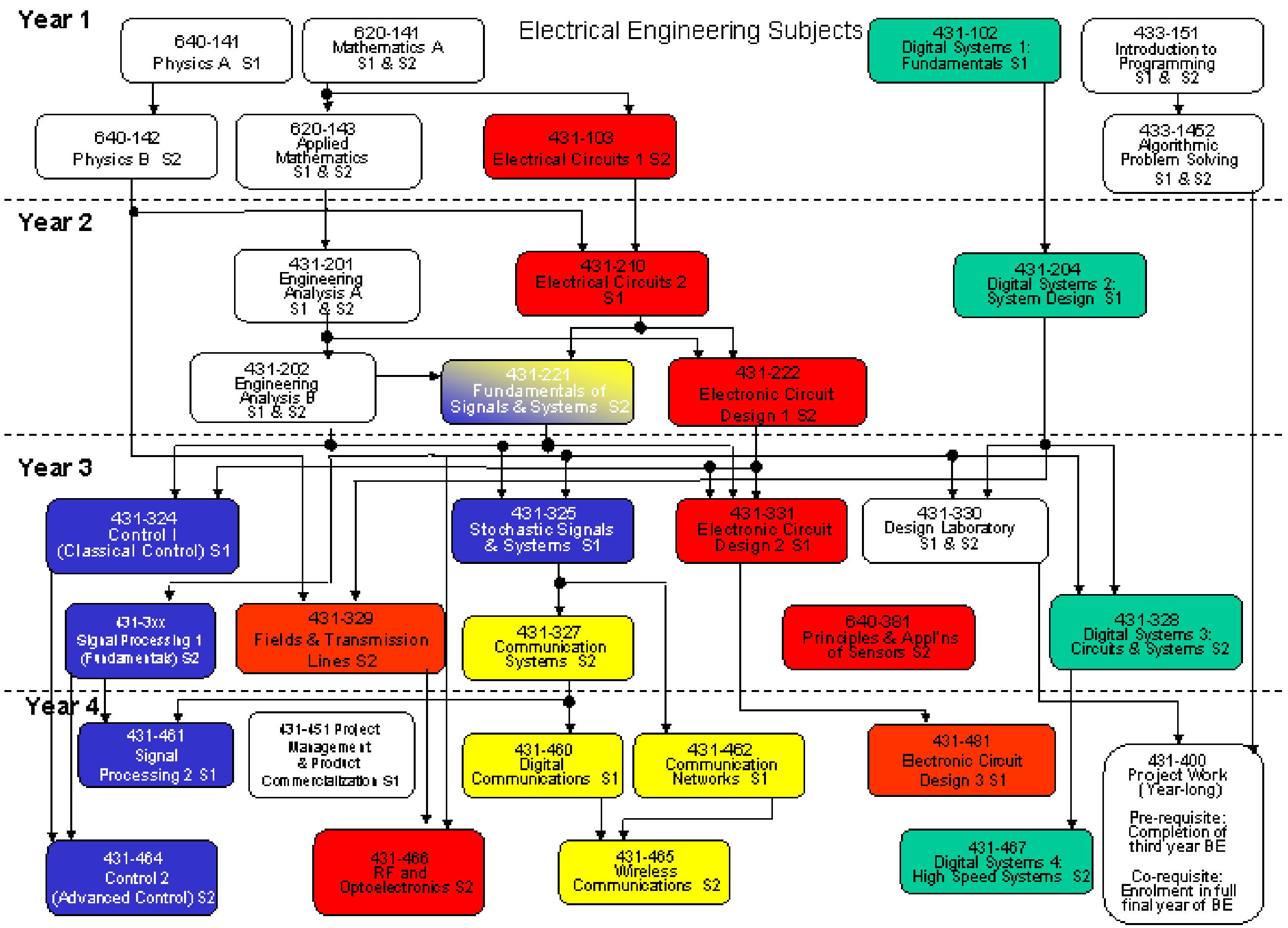
    - Flow chart of University Melbourne EE classes



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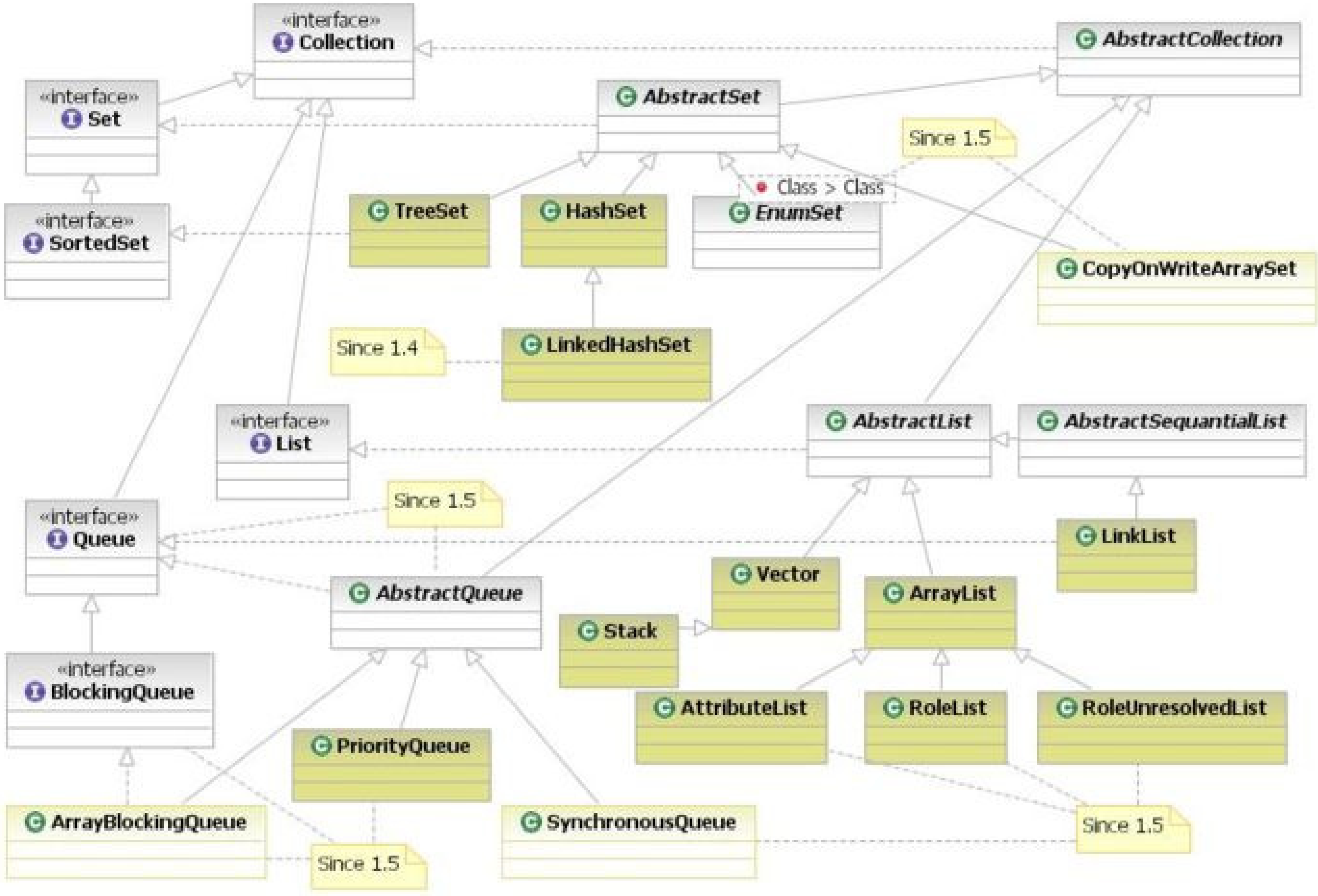

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    - Draws DAG's in a top-down manner

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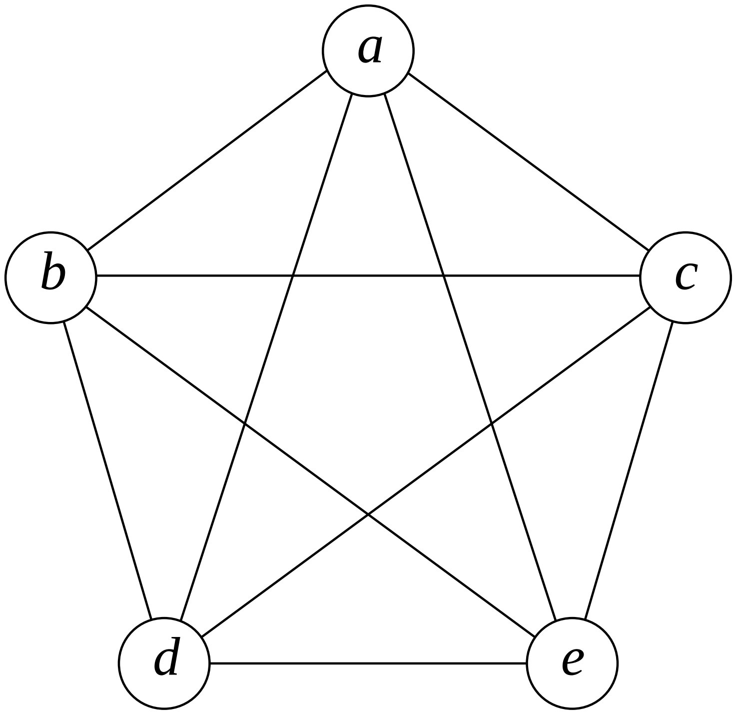
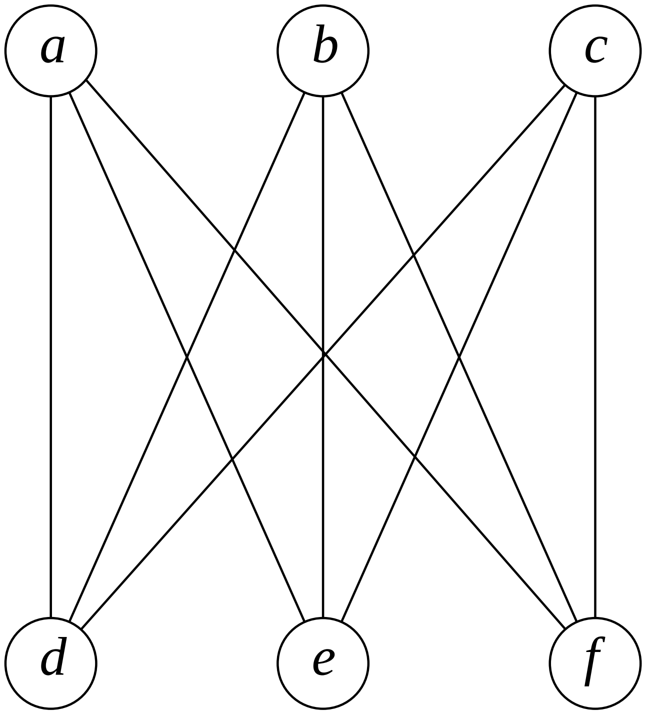


K_5



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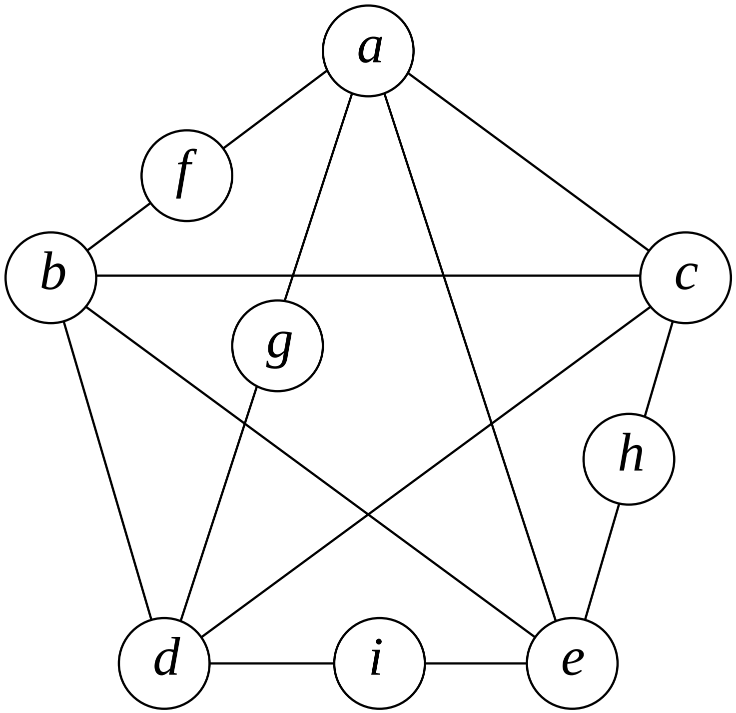
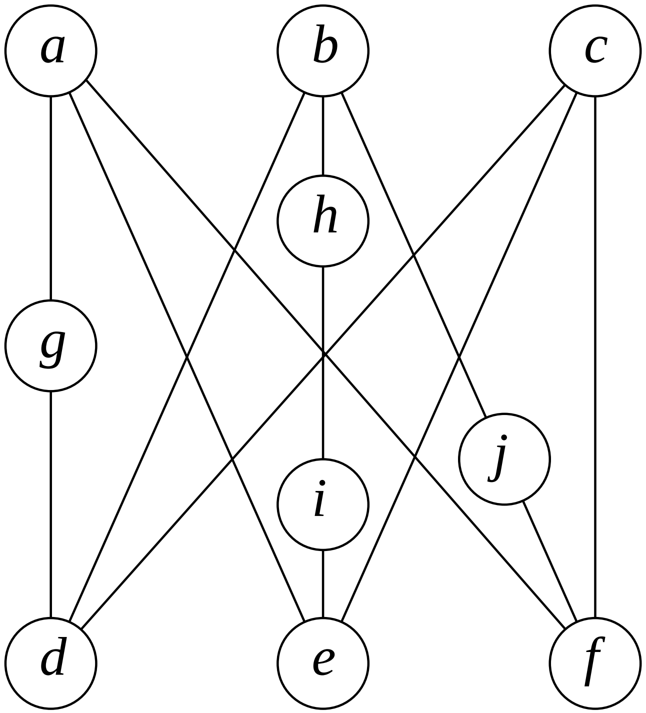


Subdivided K_5



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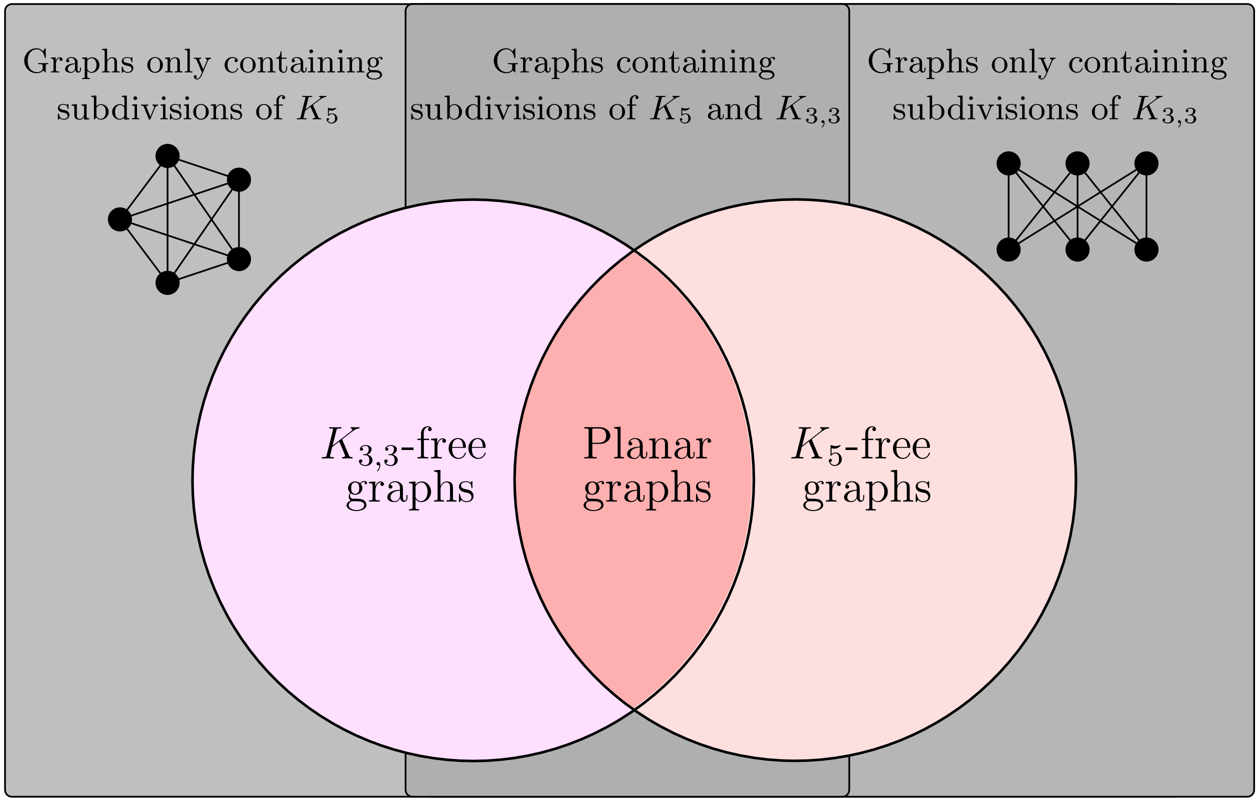

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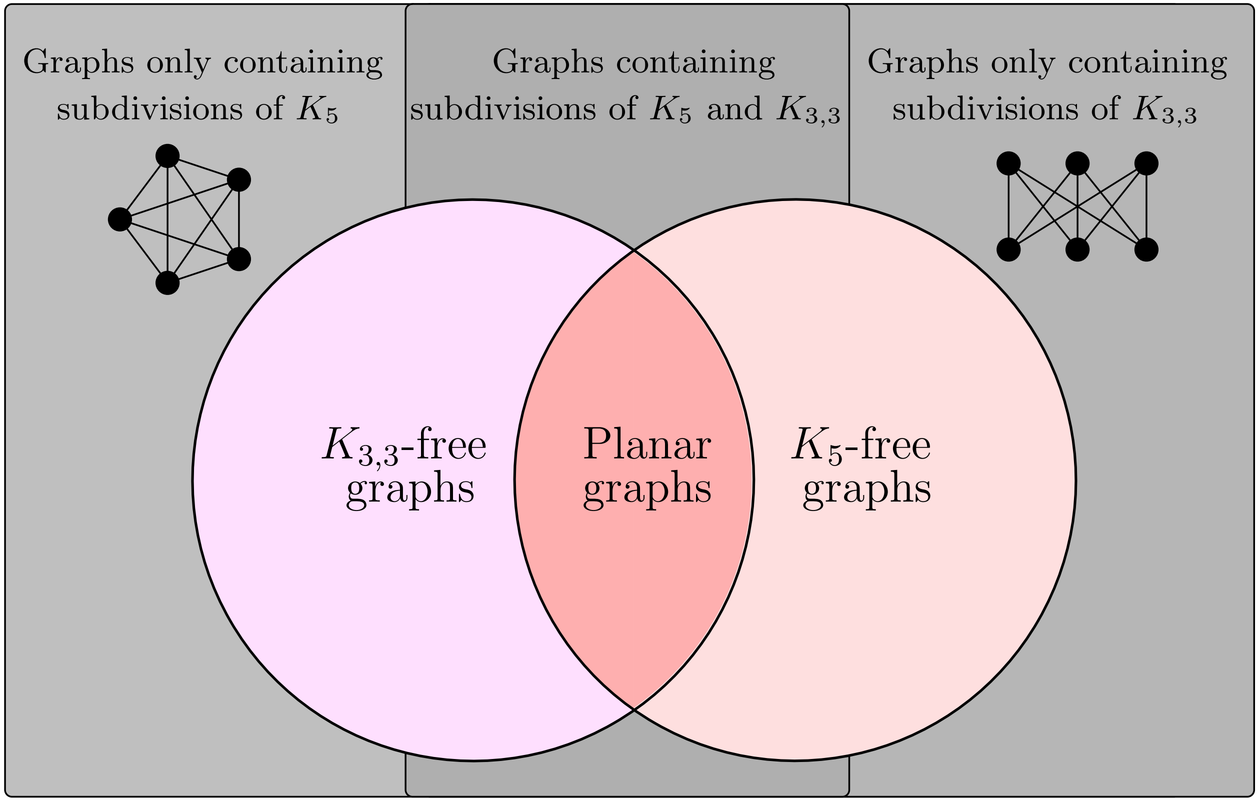

- Similar forbidden subdivision characterization for ULP graphs

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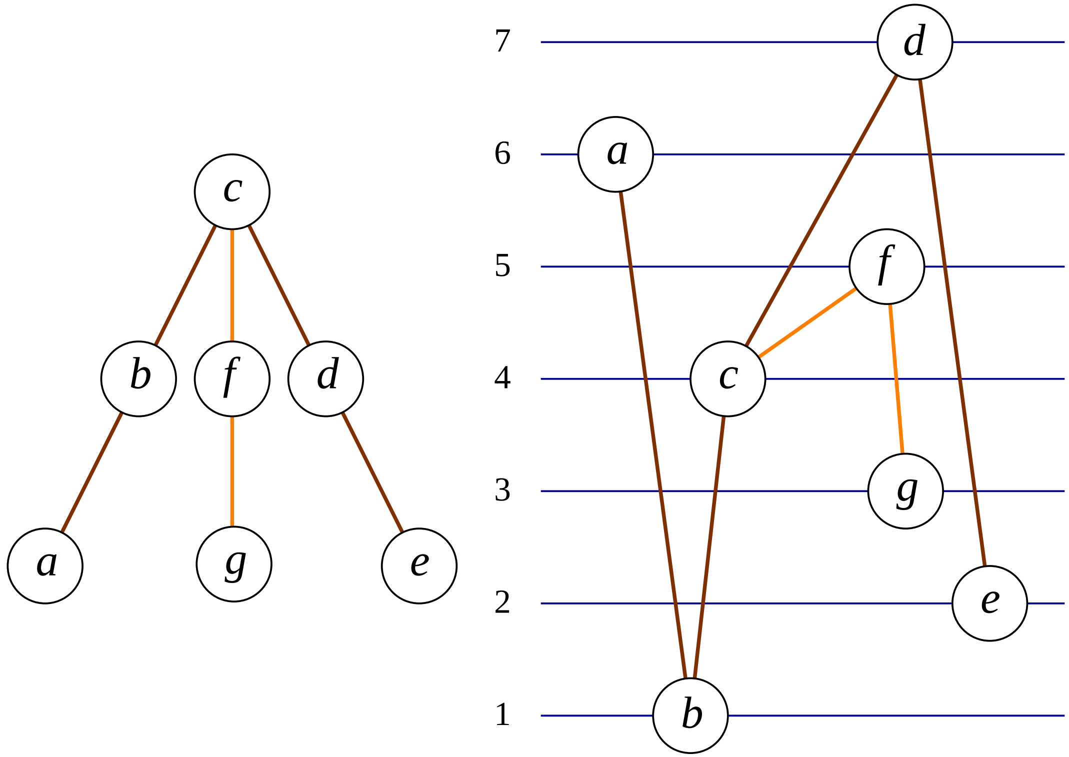

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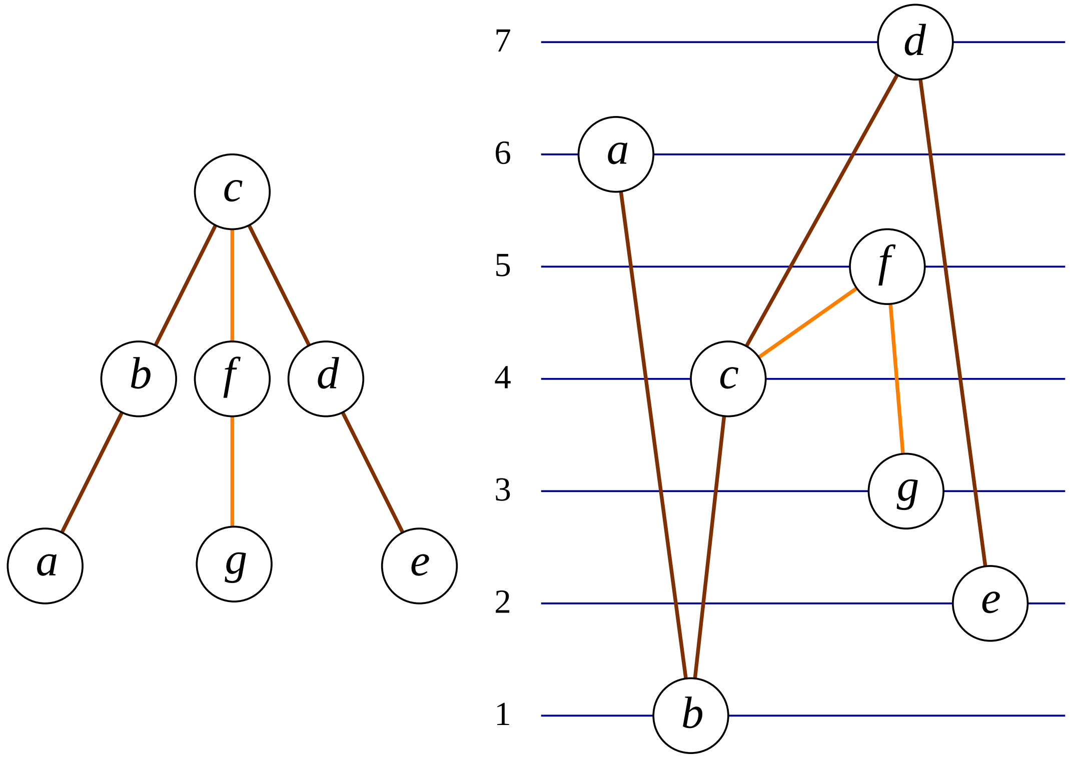

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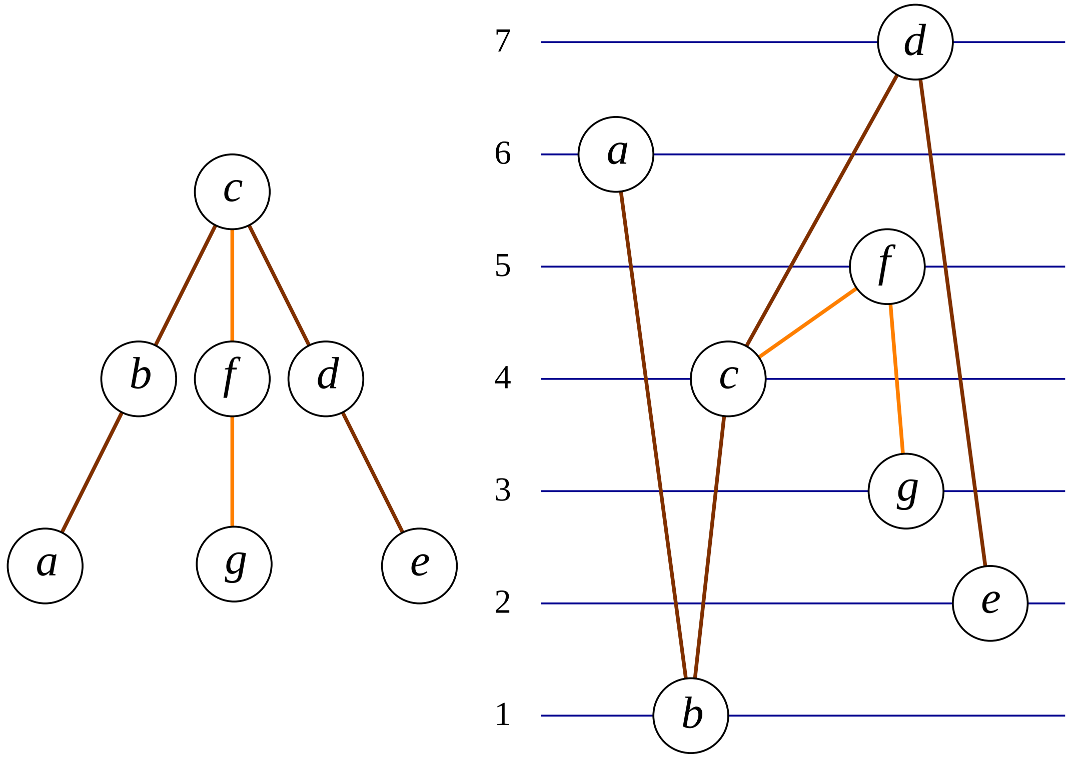

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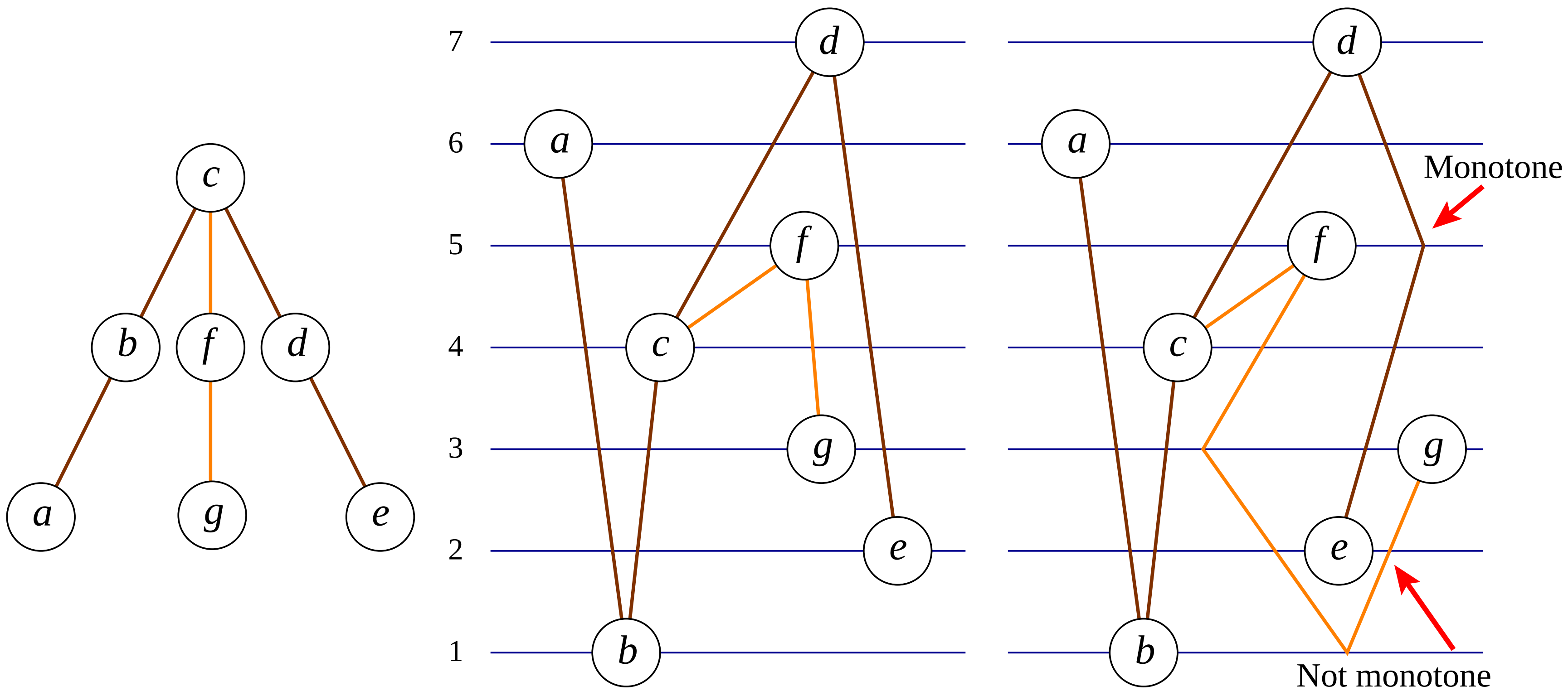

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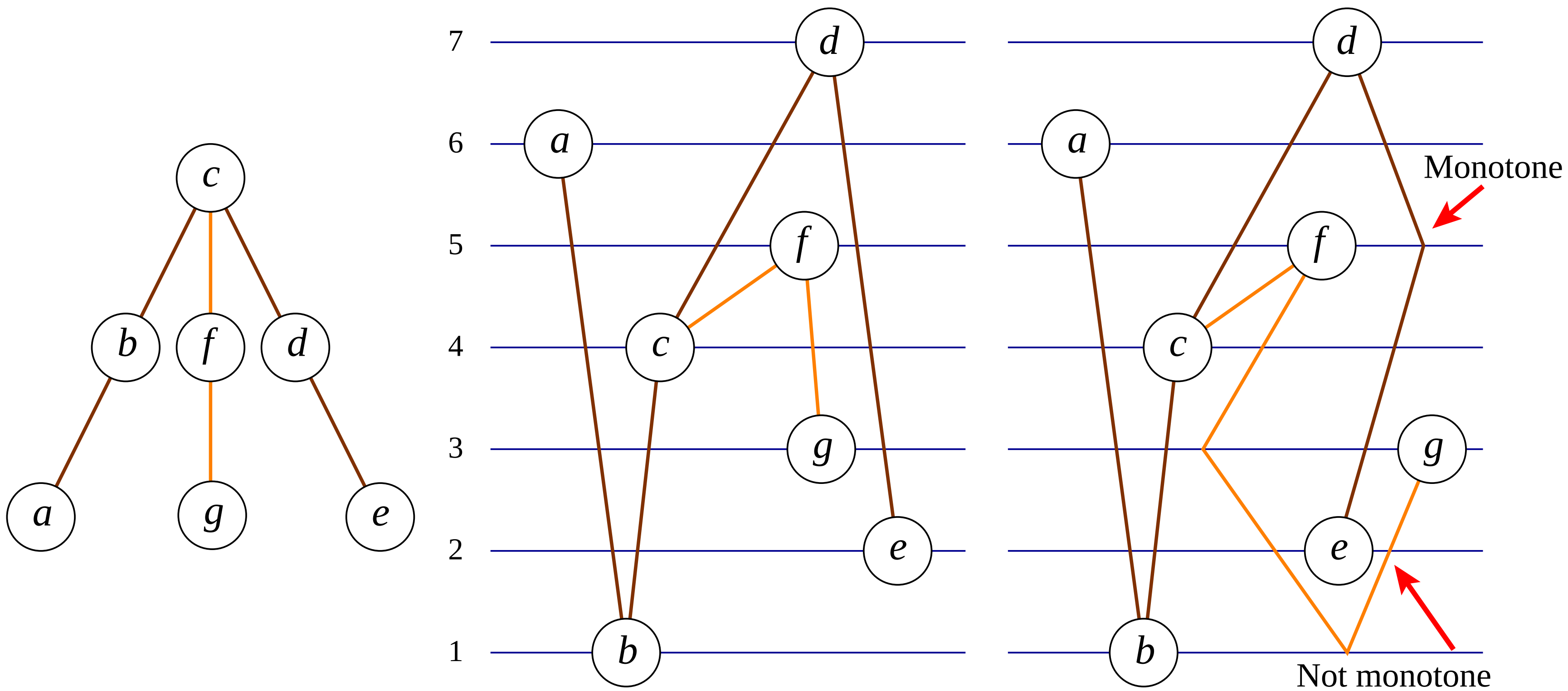

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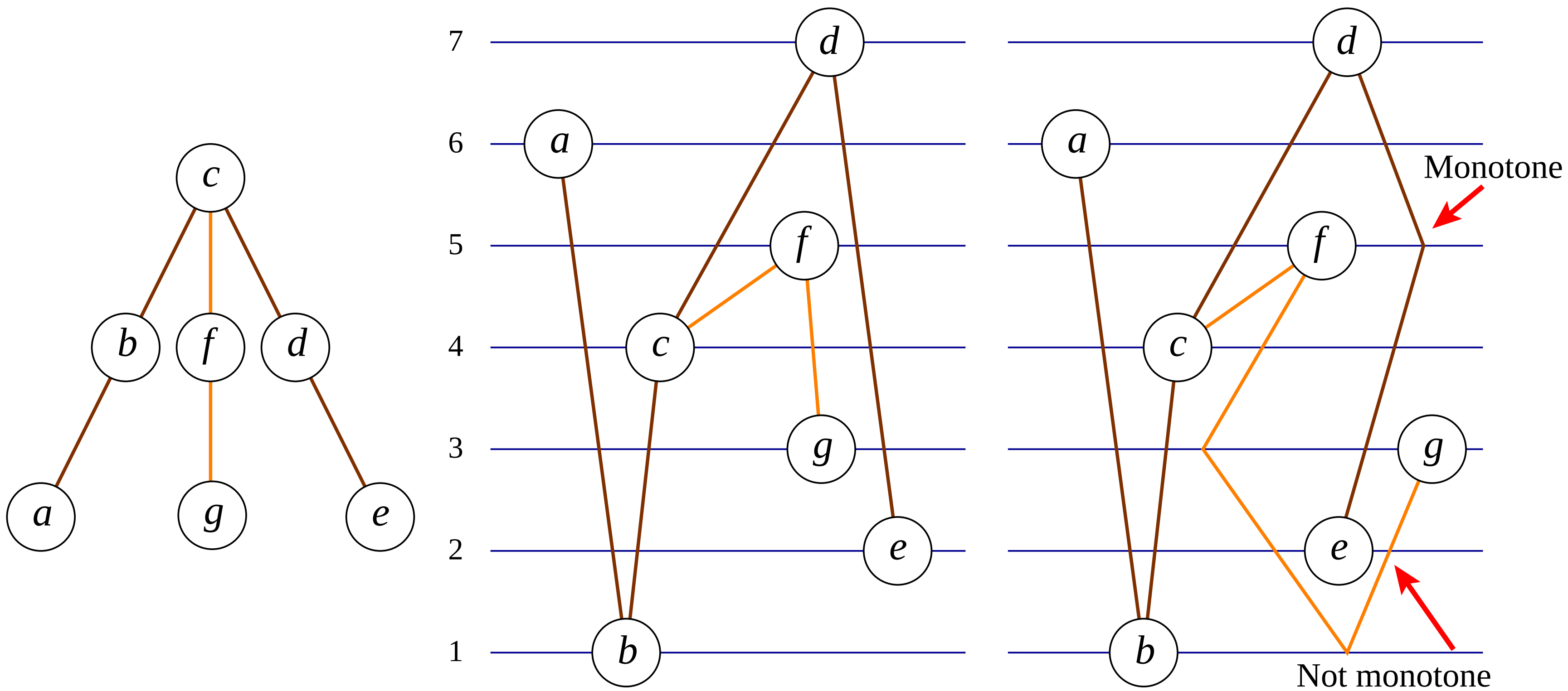

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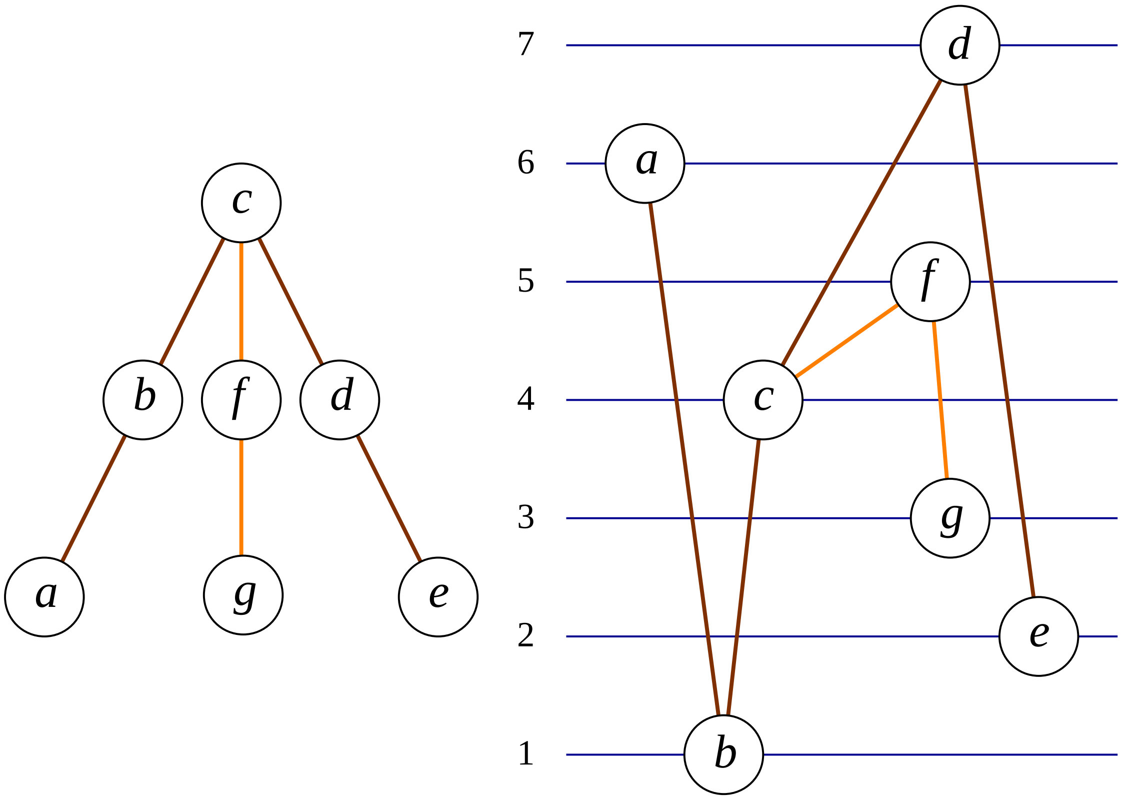
  - G can be drawn without crossings AND each vertex remains on its level

Definitions – Unlabeled Level Planarity

- Only *some* planar graphs are level planar over *every* bijective leveling



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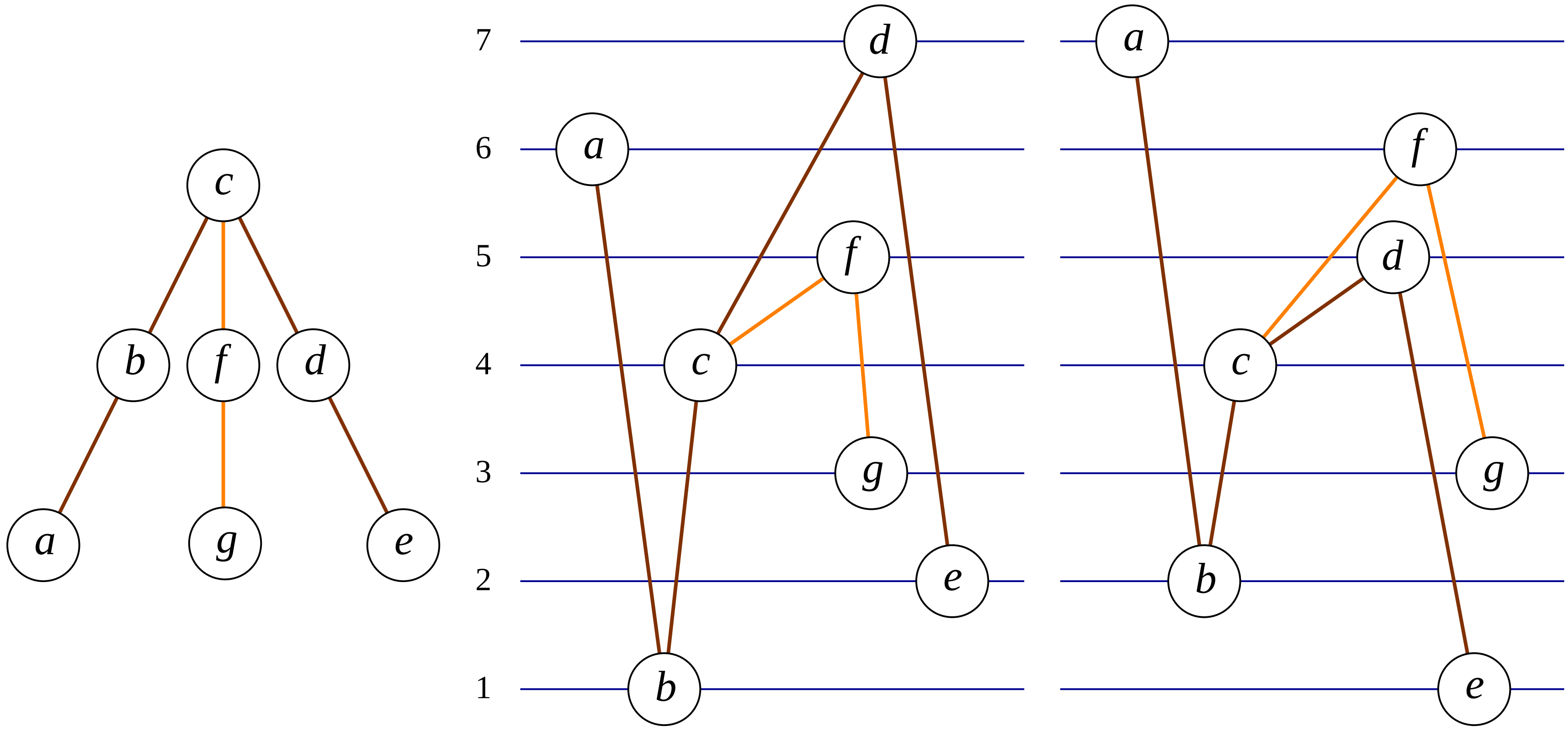

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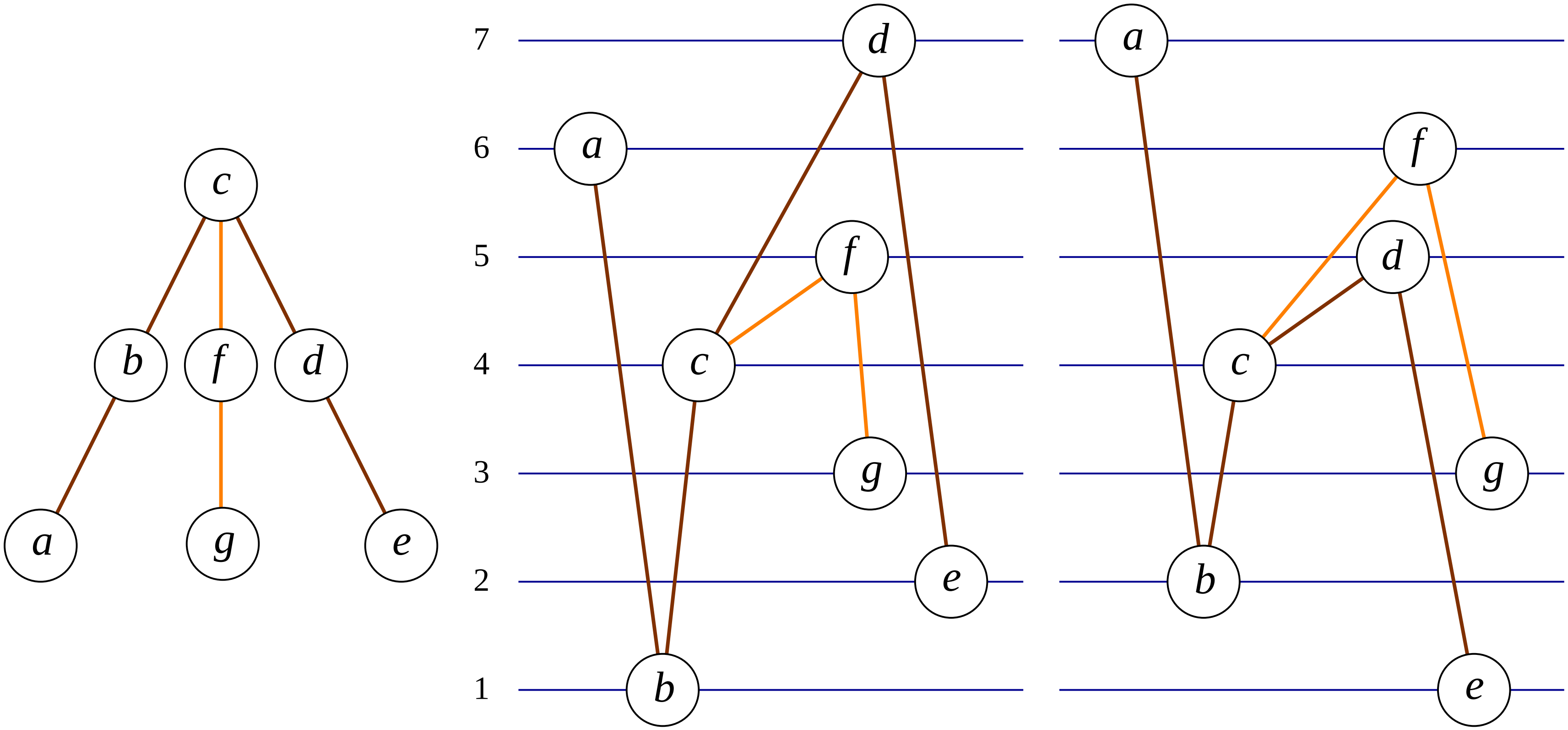
  - Such graphs are called Unlabeled Level Planar (ULP)



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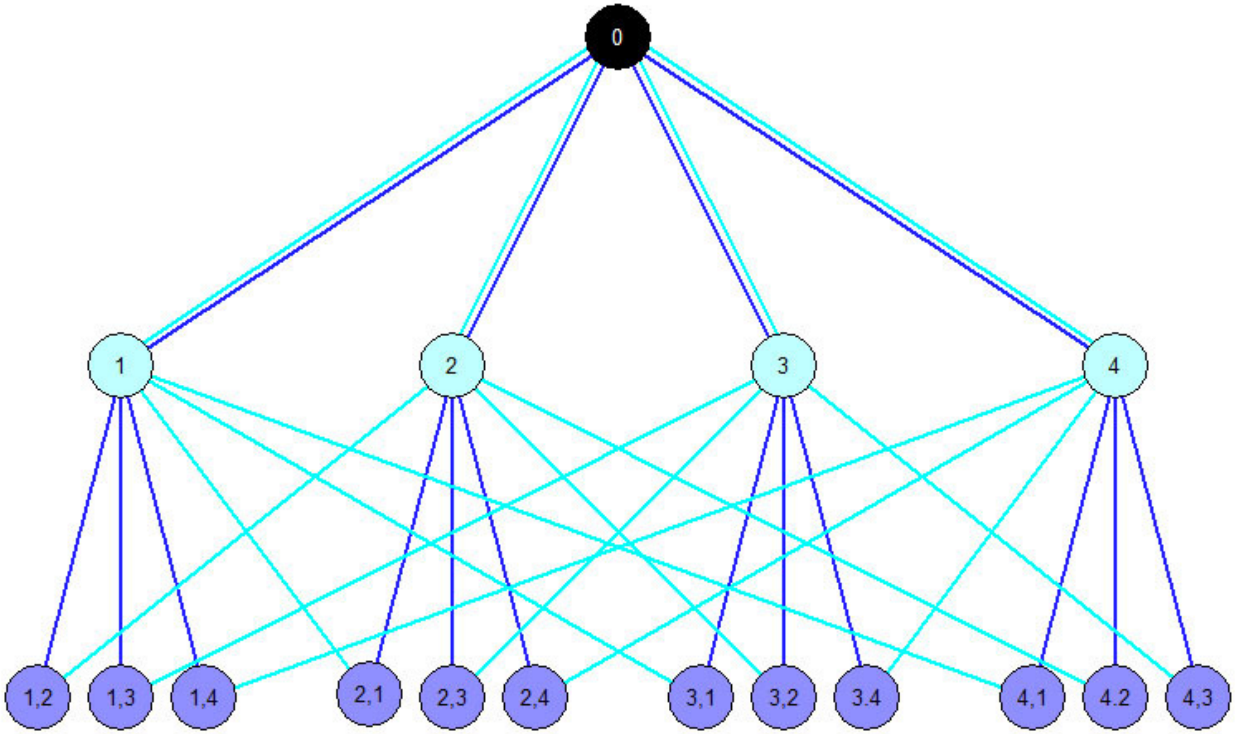
  - Generalizes the notion of planarity



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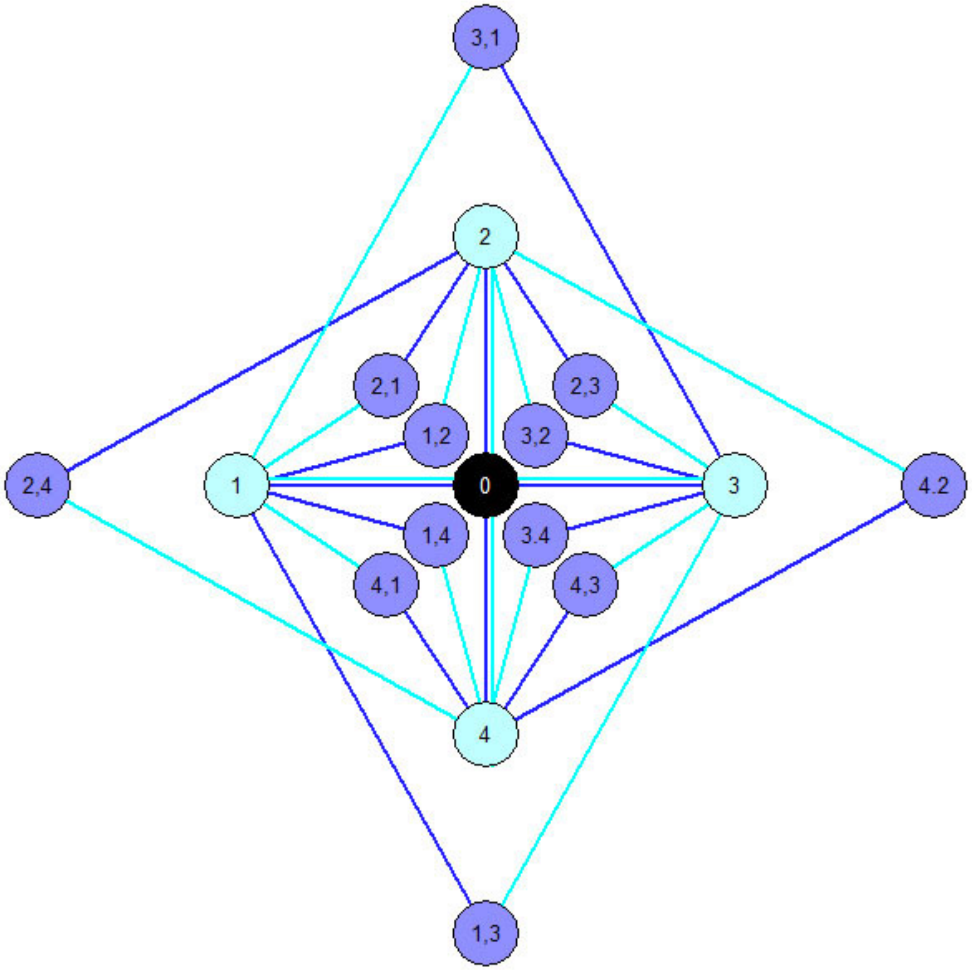
  - Related to geometric thickness



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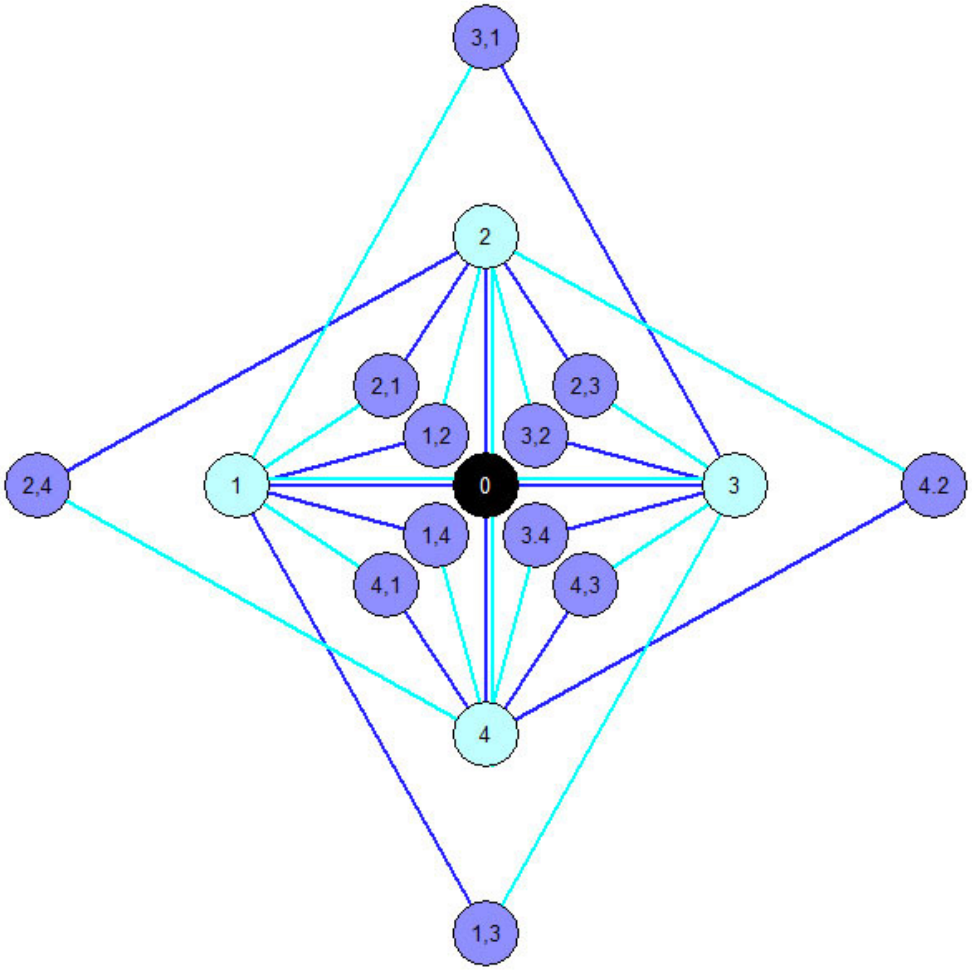

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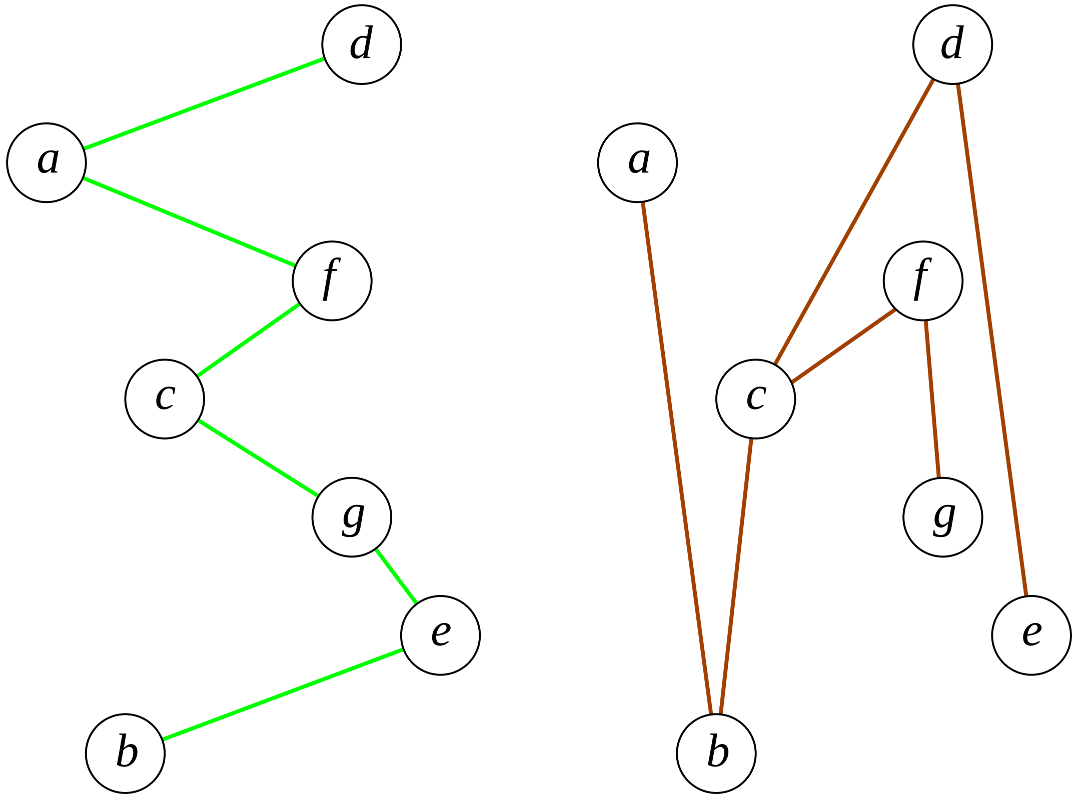
- Simultaneously embed monotone path with any ULP graph
  - Mapping between vertices is given by labeling



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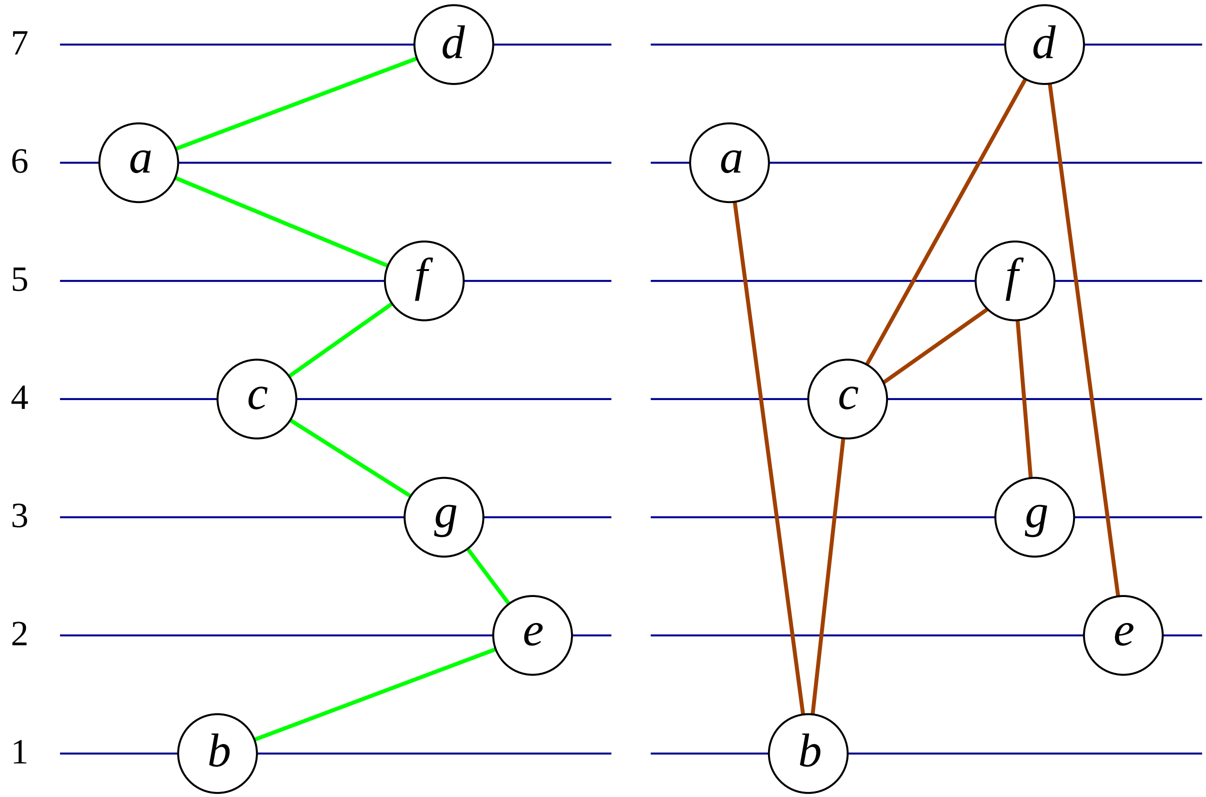
  - Ordering of the vertices in path gives a leveling



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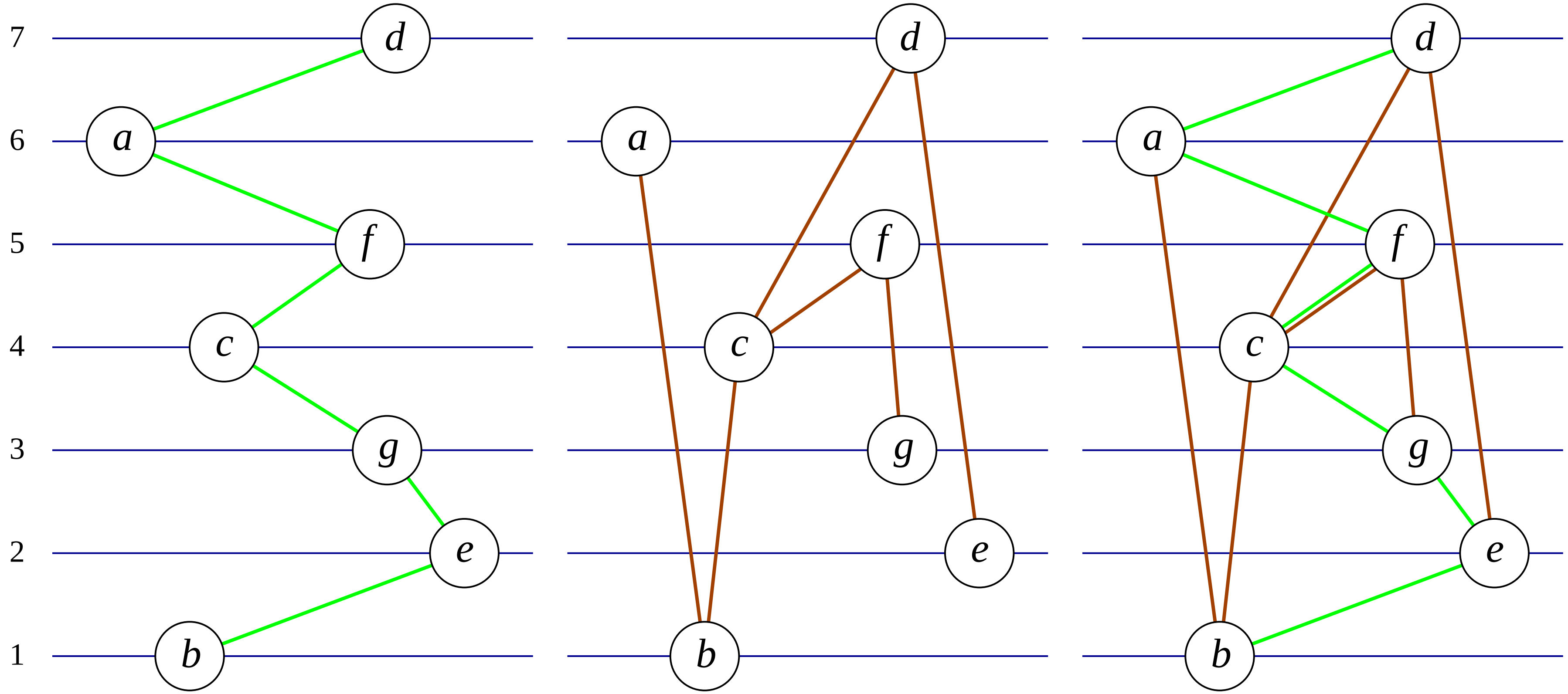
  - Graphs overlaid so that vertices with same label have same position



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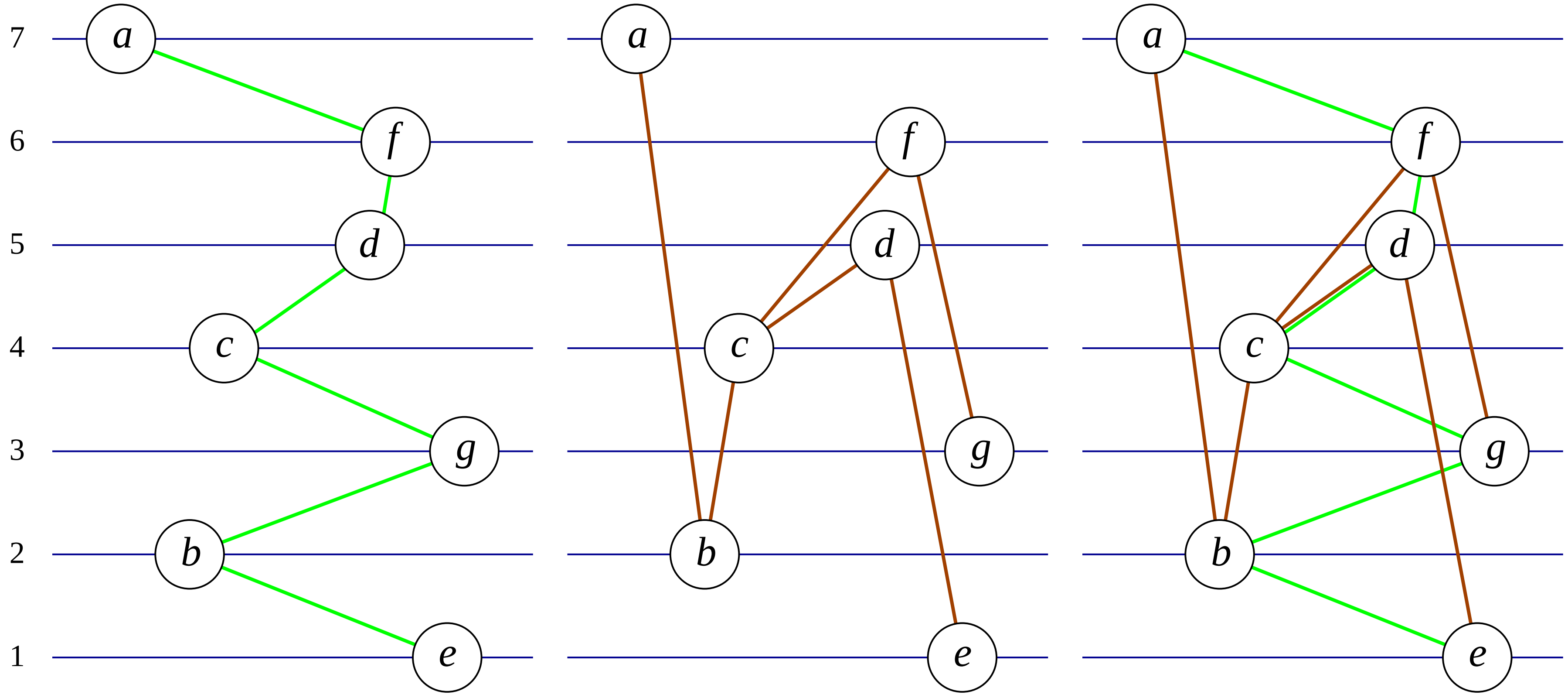
  - Has to work for *any* of the $n!$ labelings between graphs



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    - A *level embedding* is the left-to-right ordering of vertices along a level

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    - Shows any level planar graph drawn with bends can be drawn with straight-line edges instead

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  - Di Battista and Nardelli characterized hierarchies in 1988
    - Uses level non-planar (LNP) patterns

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  - Di Battista and Nardelli characterized hierarchies in 1988
  - Healy, Kuusik, and Leipert found minimal LNP subgraph patterns in 2000
    - Incomplete – do not match all forbidden ULP graphs

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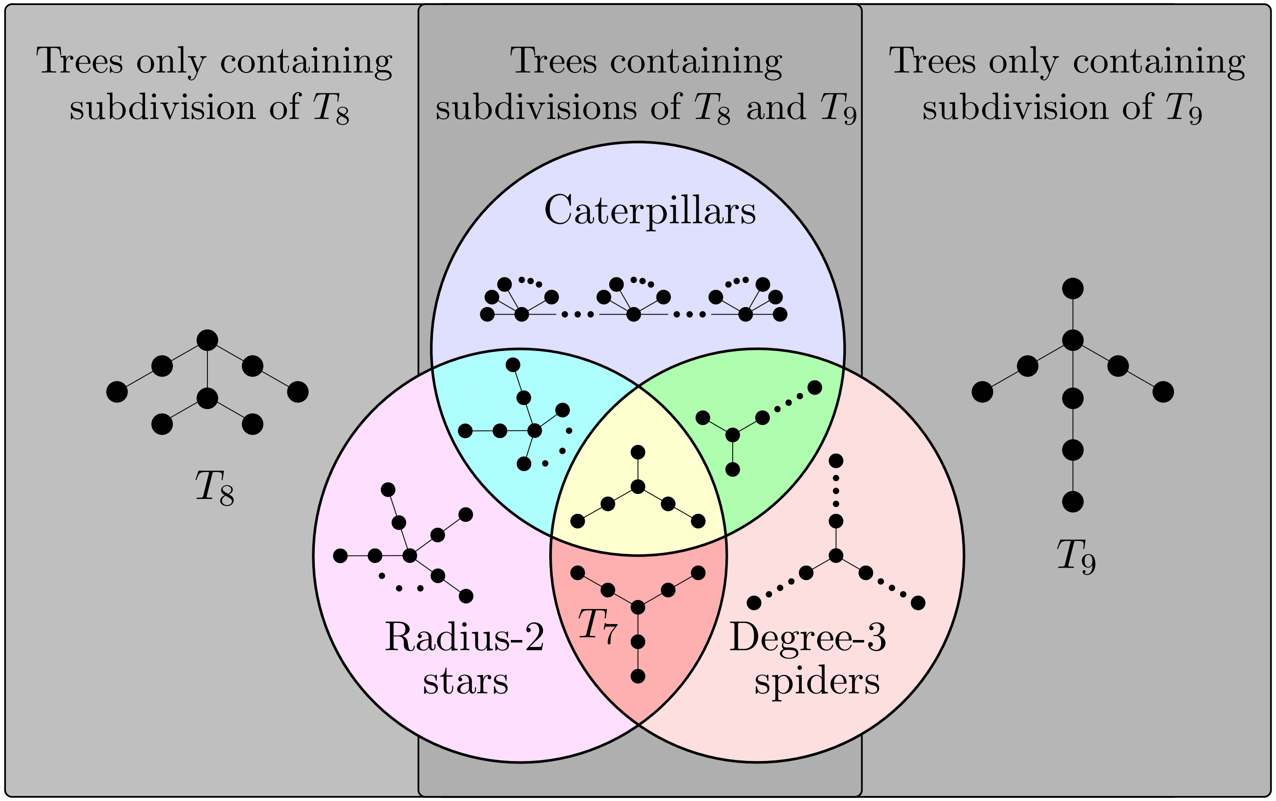

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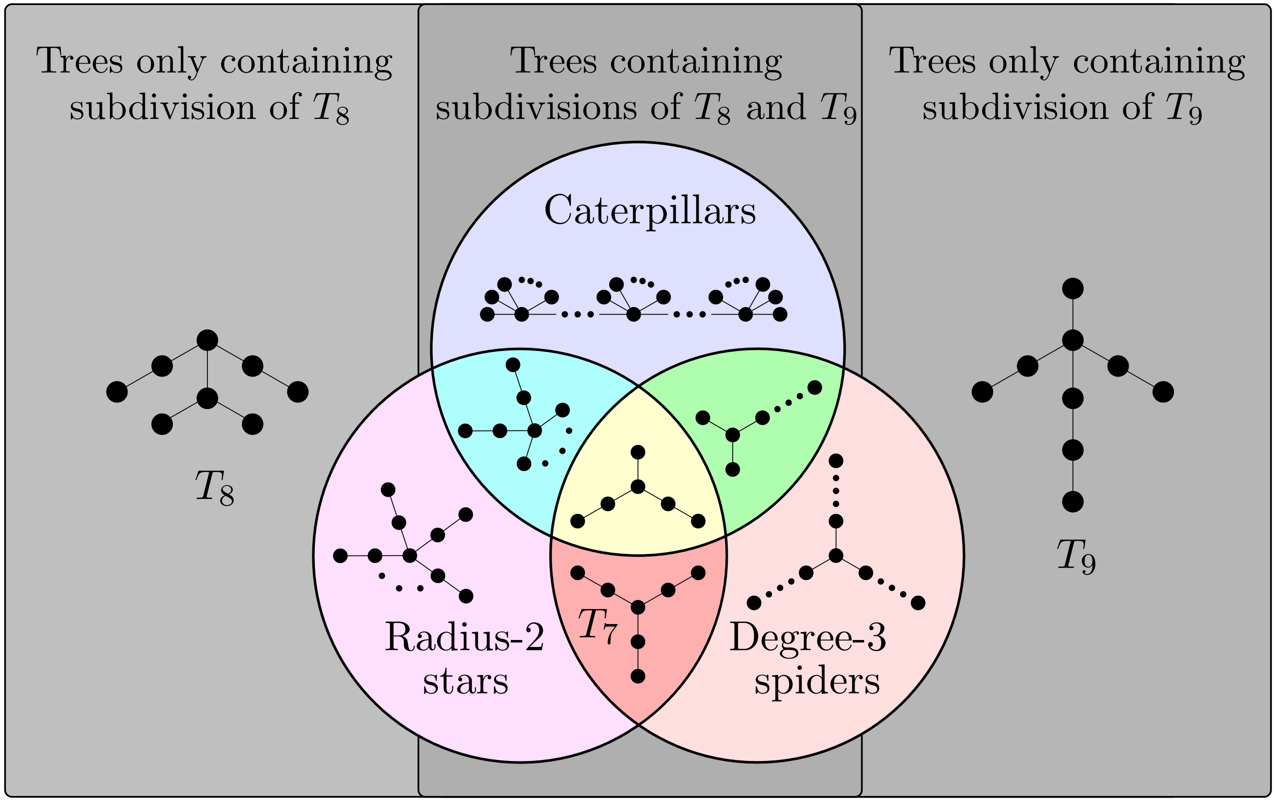

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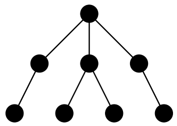

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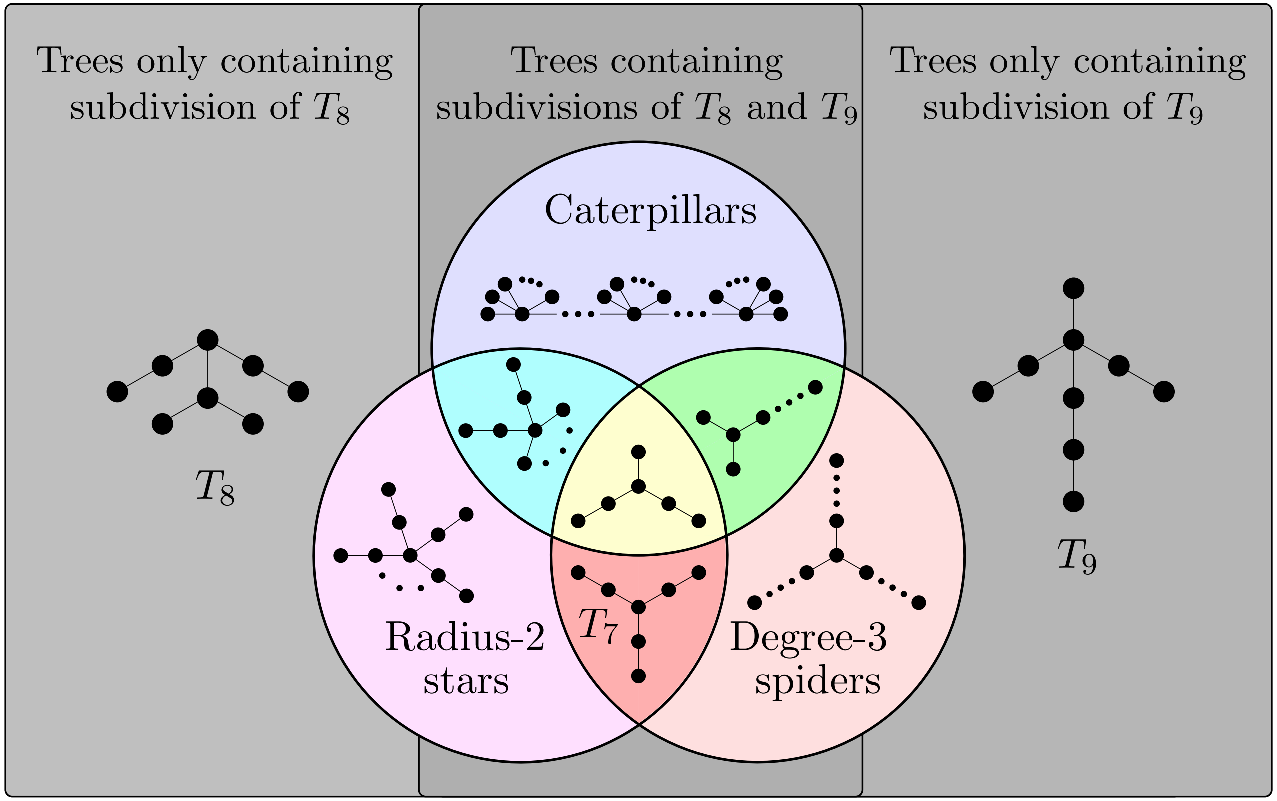

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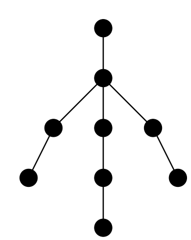
  - Tree T_9 with 9 vertices and one node of degree 4



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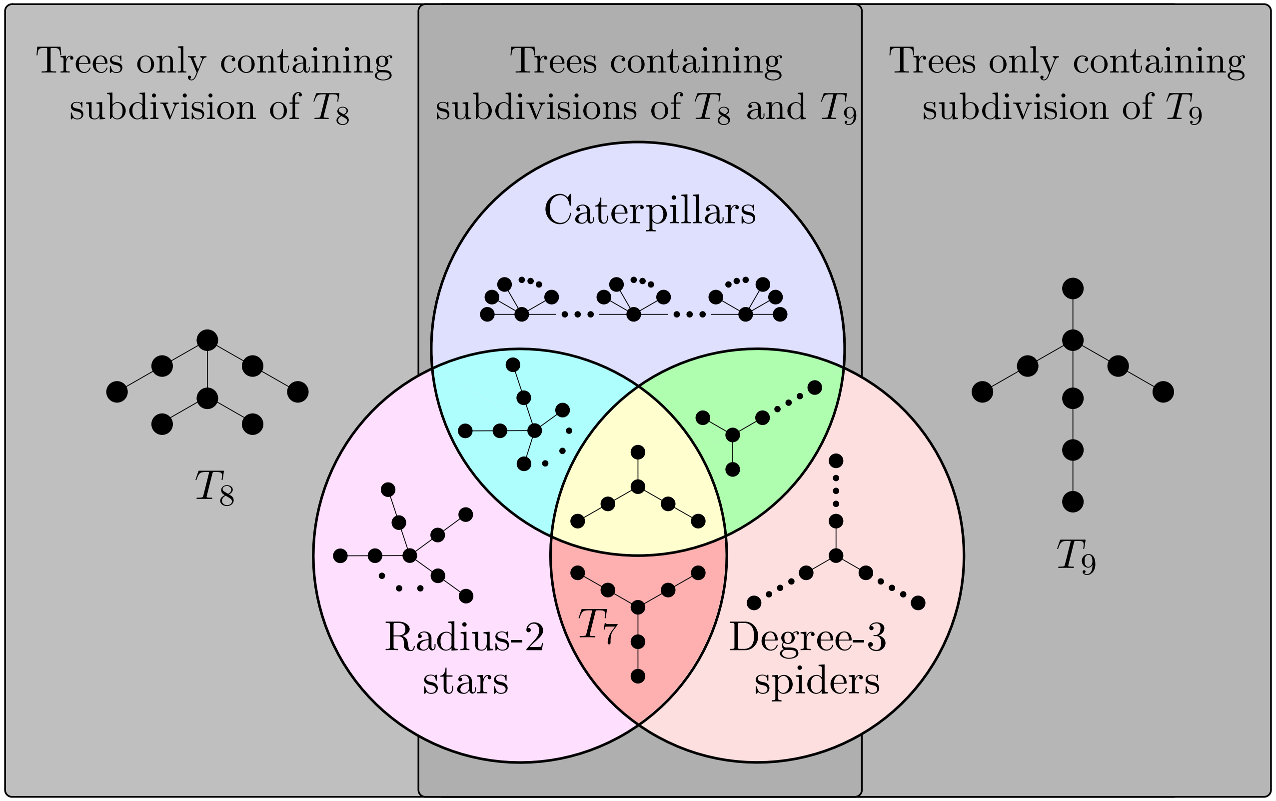

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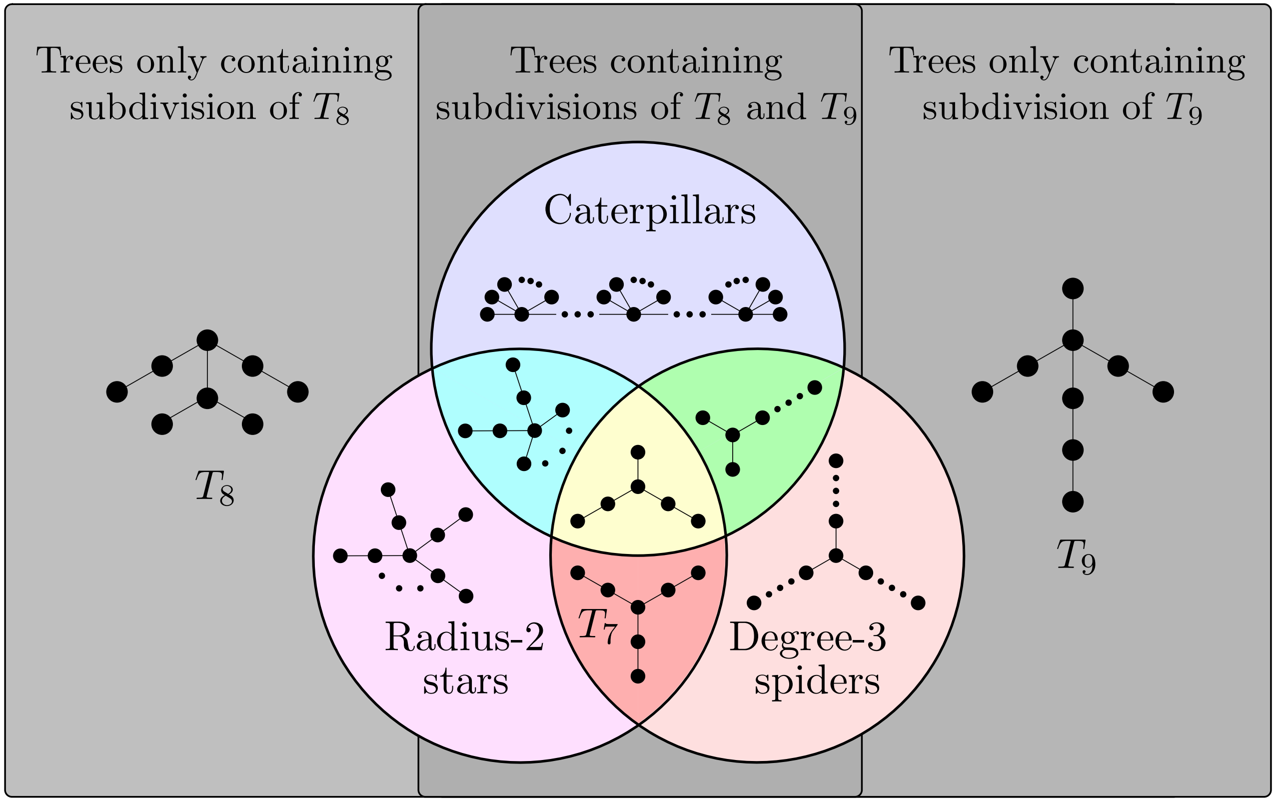

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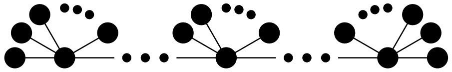

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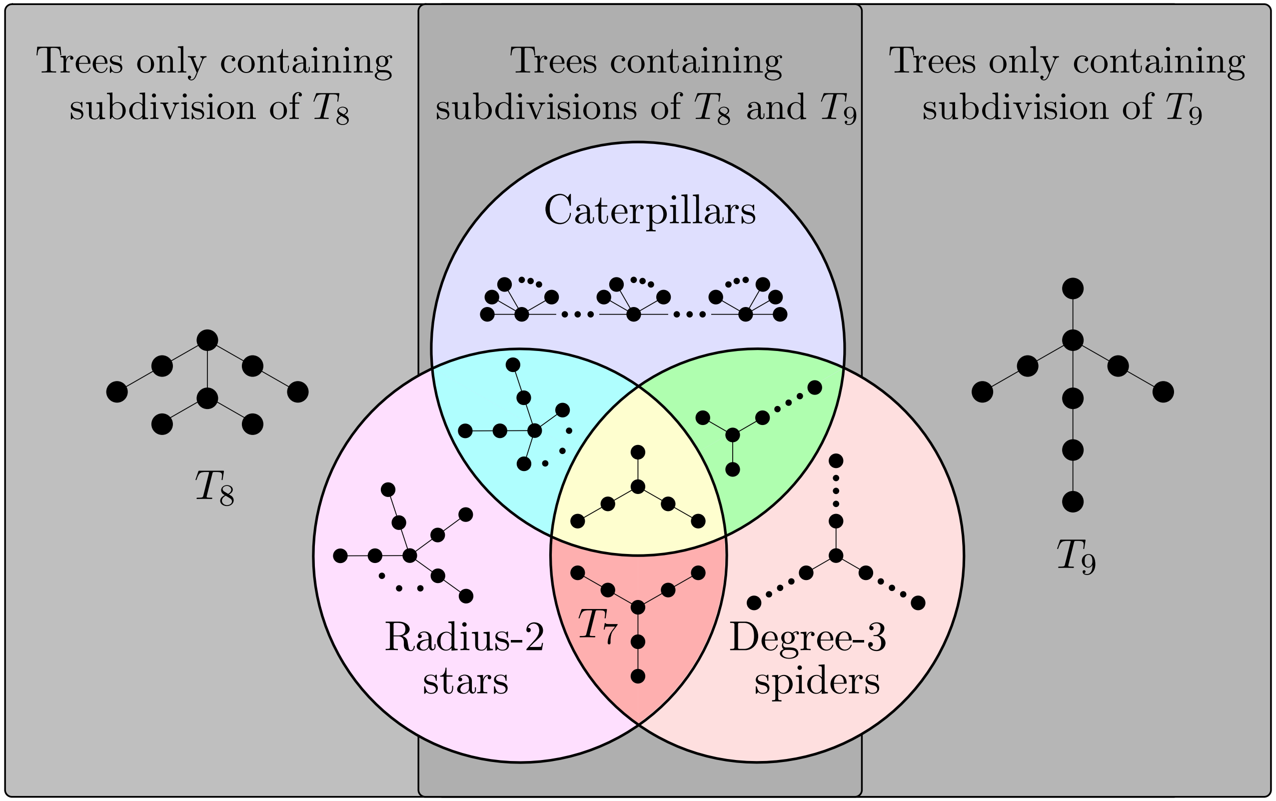

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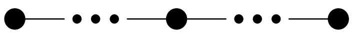

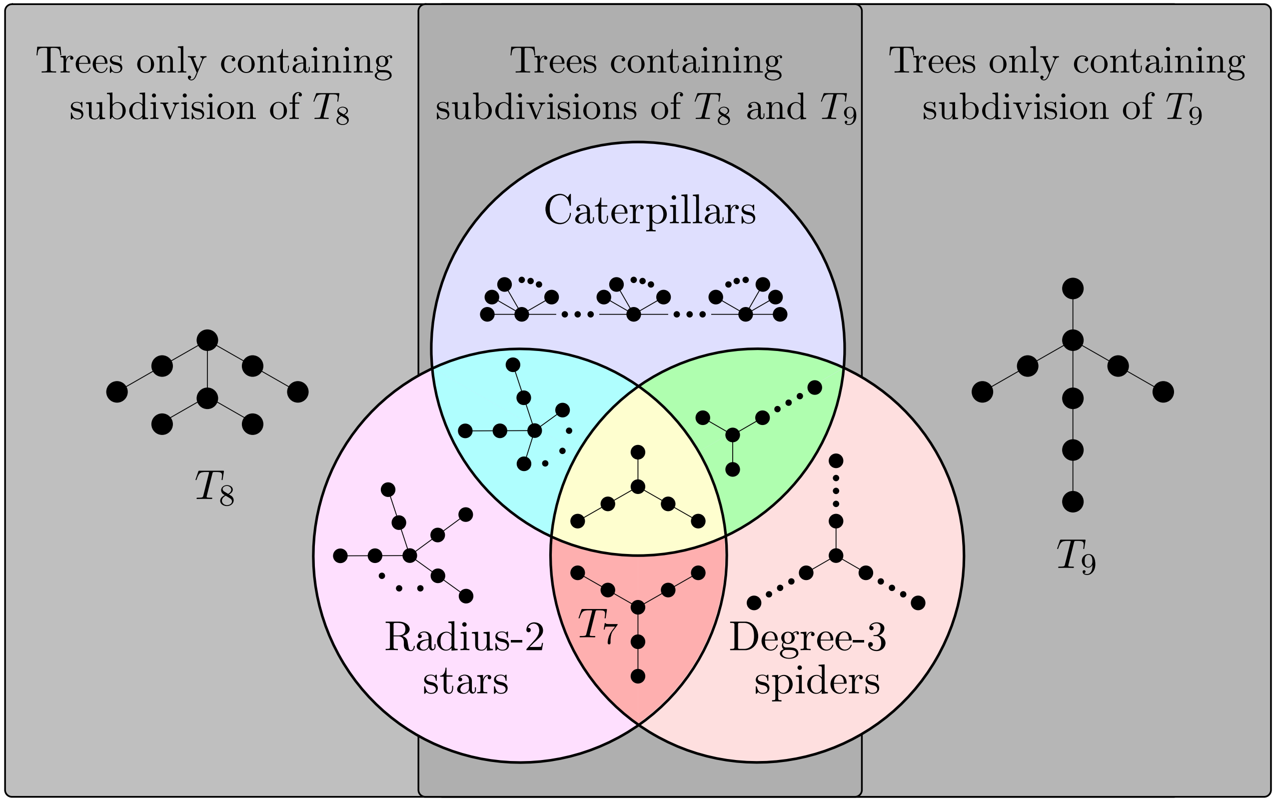

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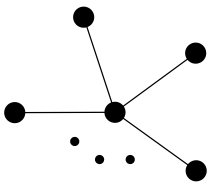

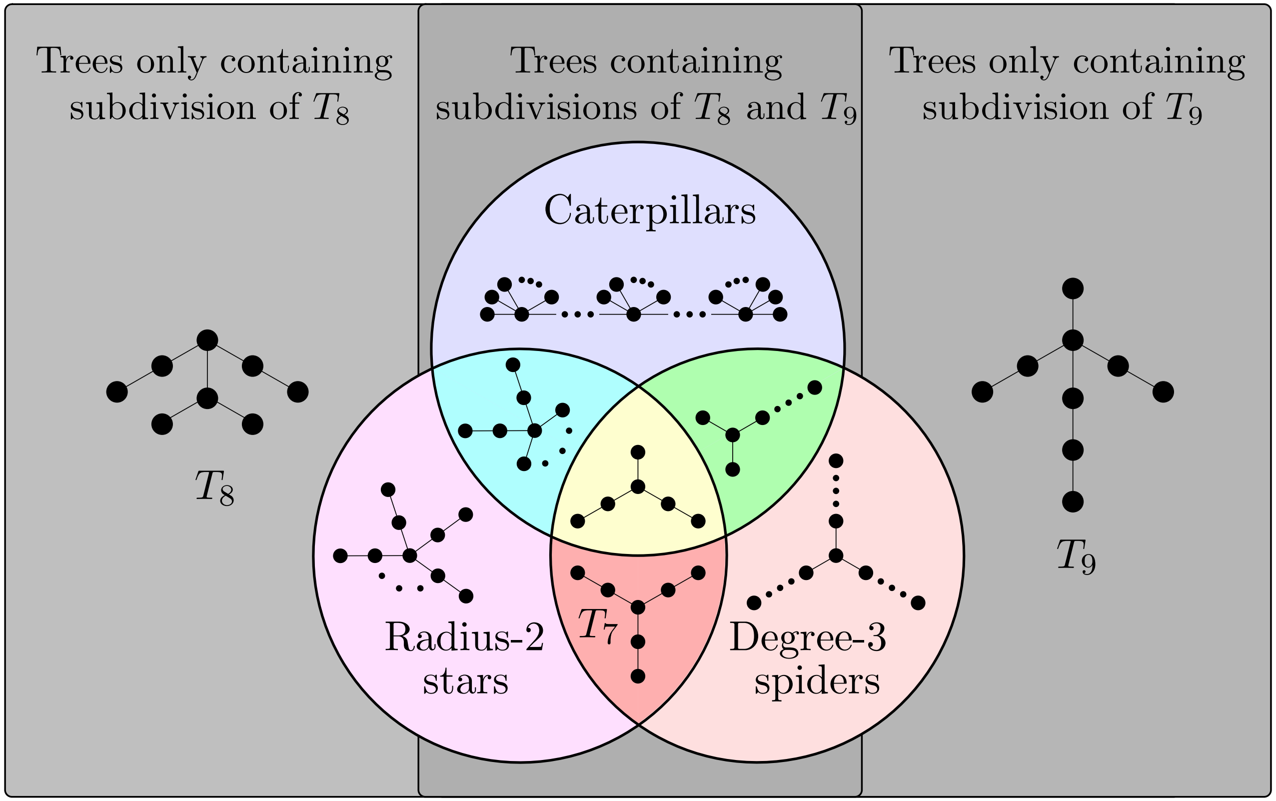

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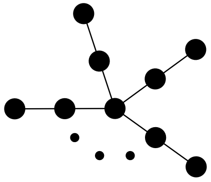

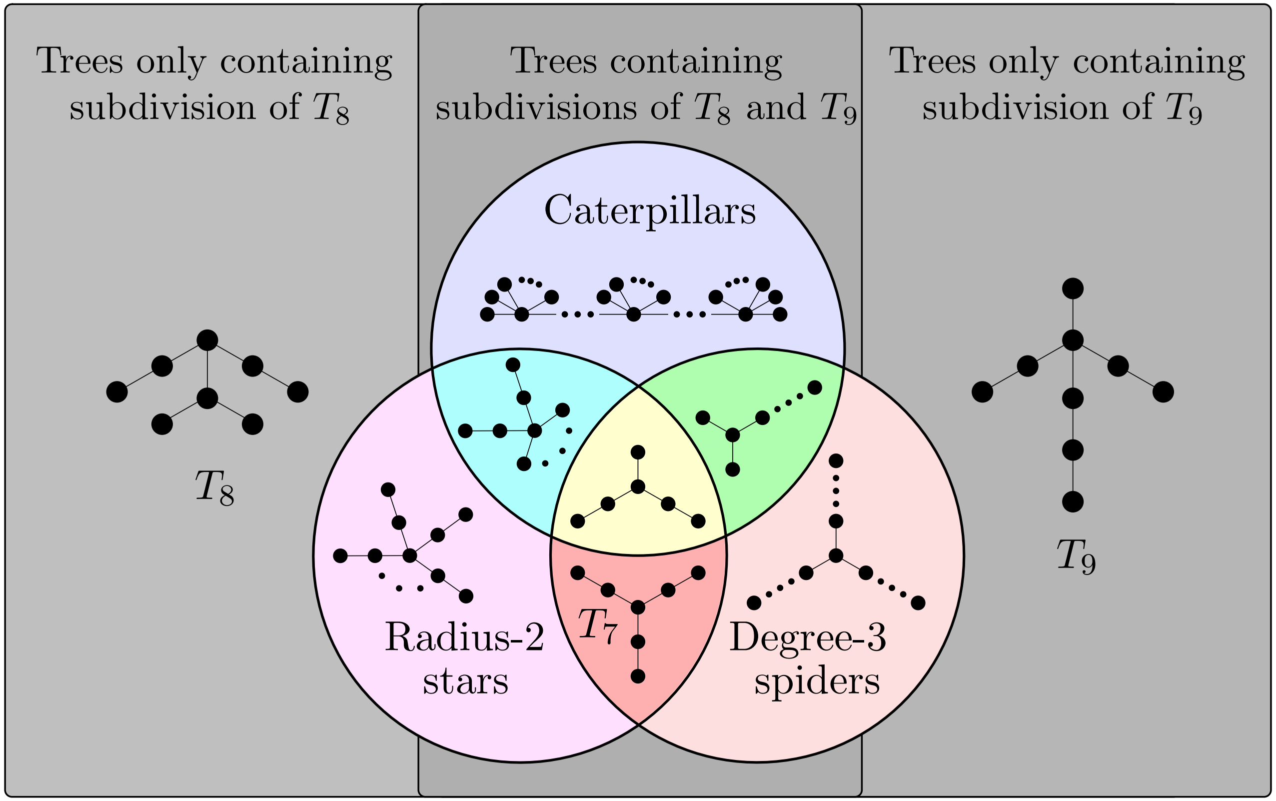

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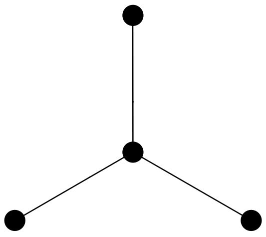

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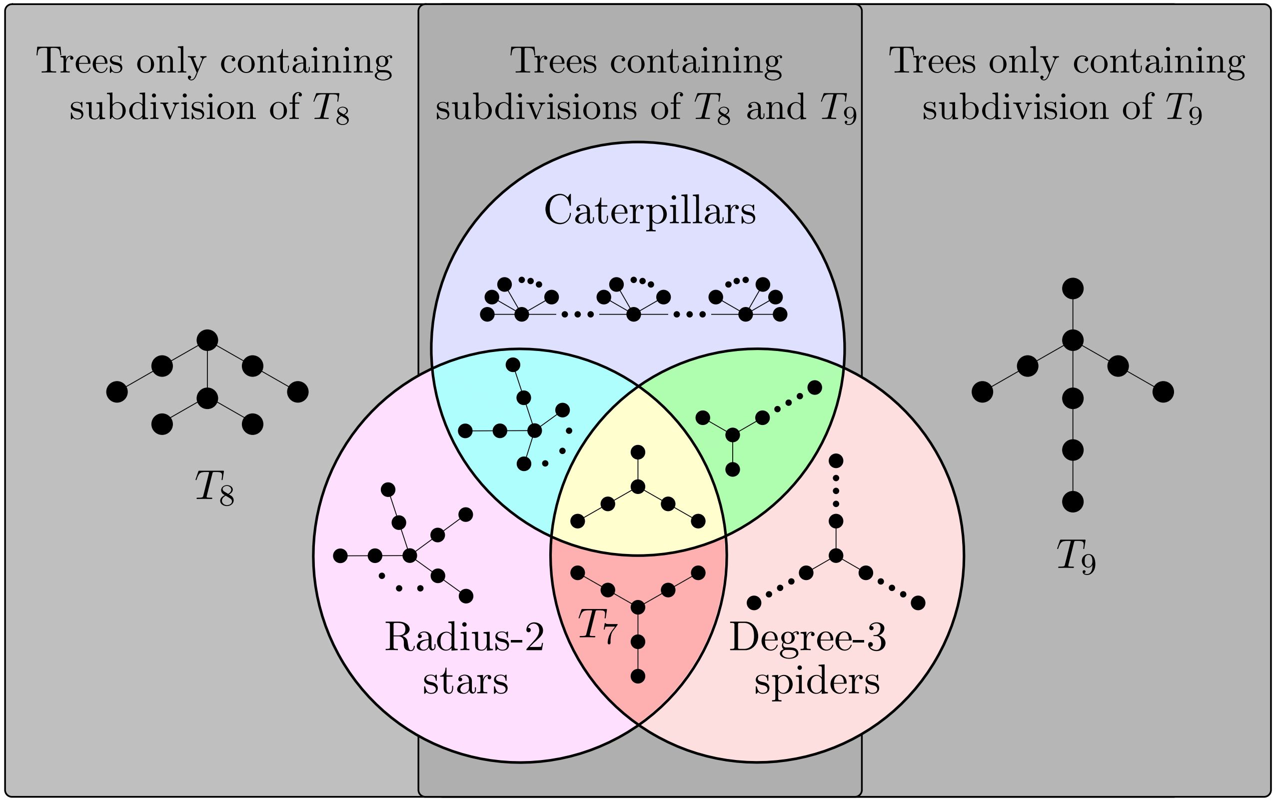

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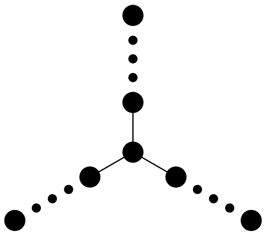
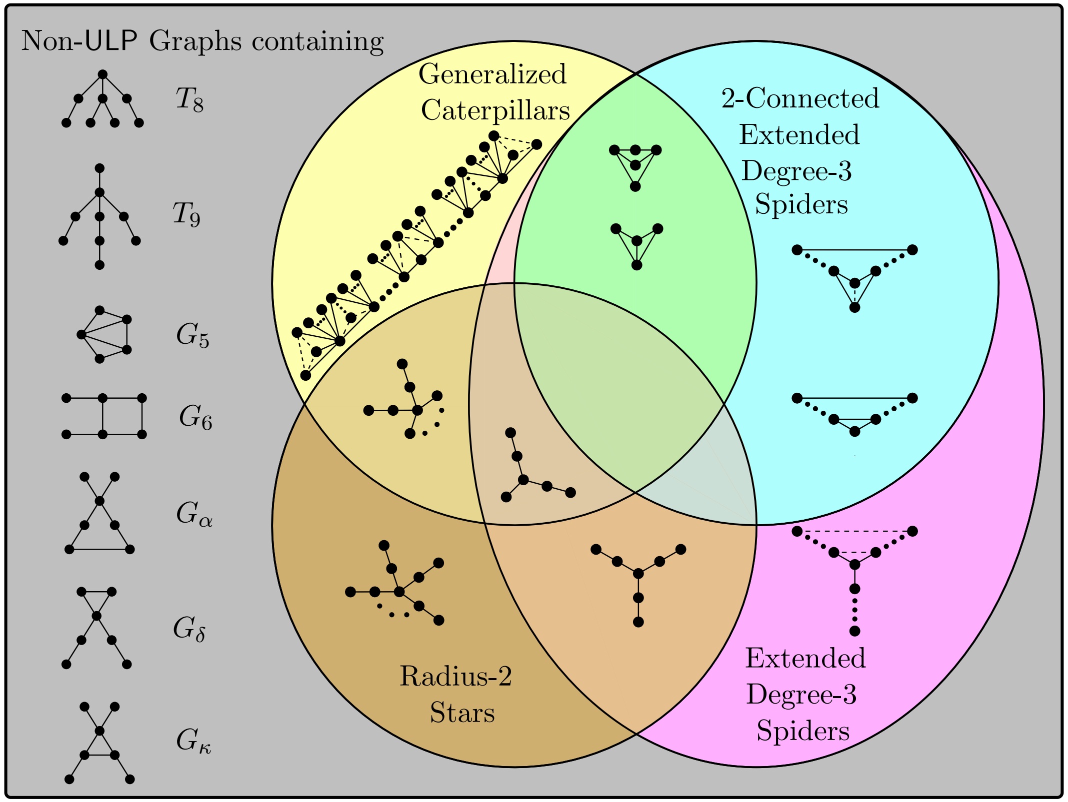
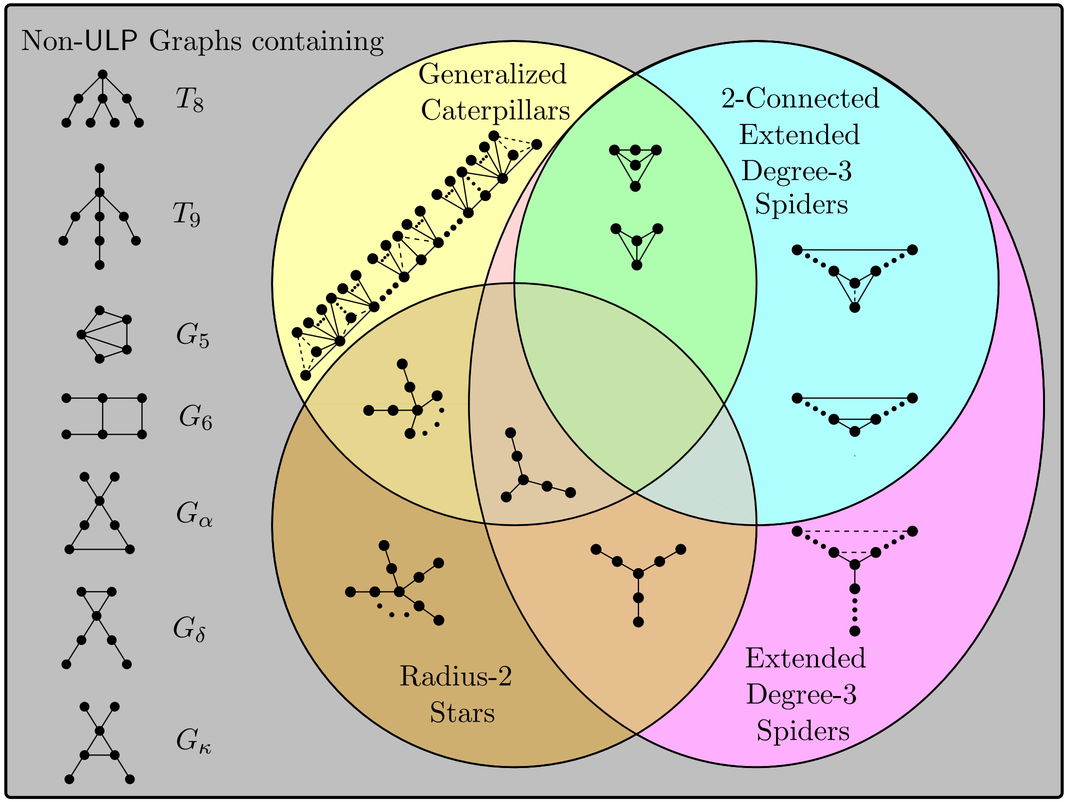
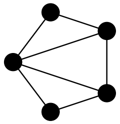


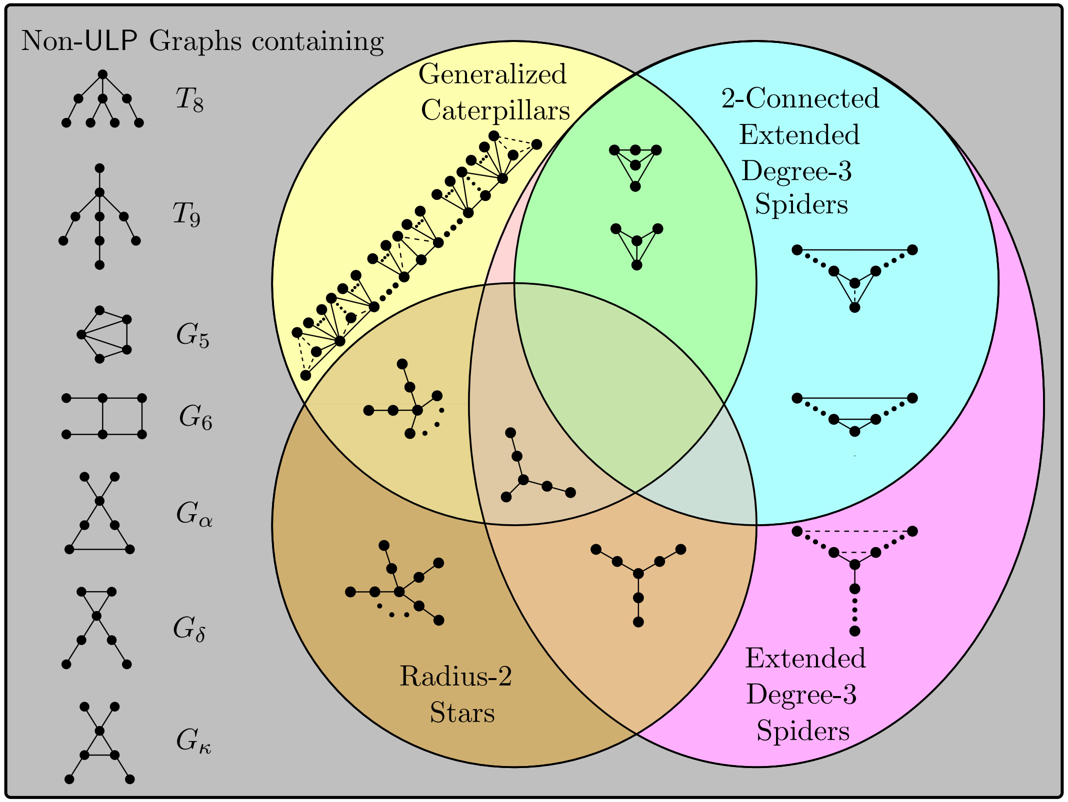
New Results – ULP Graphs



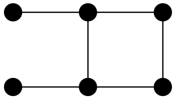
- Characterization of ULP graphs by 7 forbidden subdivisions



- Characterization of ULP graphs by 7 forbidden subdivisions
  - Graph G_5 with 5 vertices and three nodes of degree 4 



- Characterization of ULP graphs by 7 forbidden subdivisions
  - Graph G_6 with 6 vertices and two nodes of degree 3



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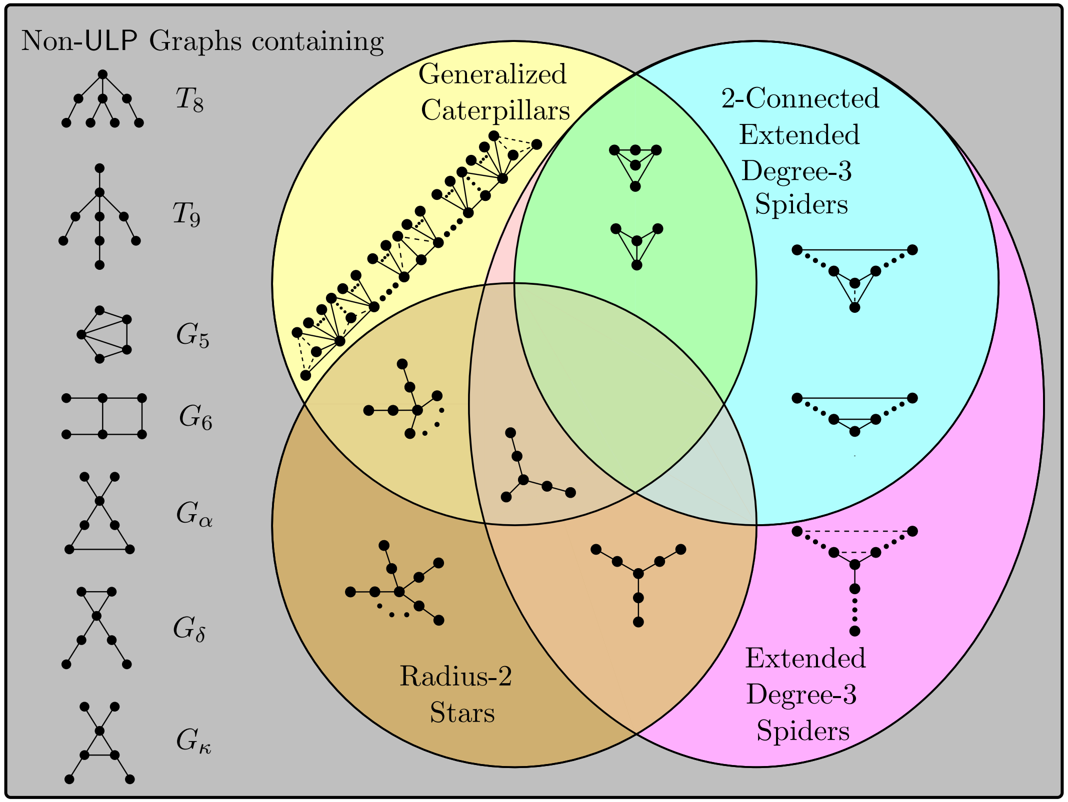
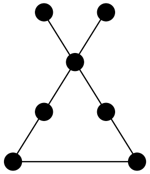
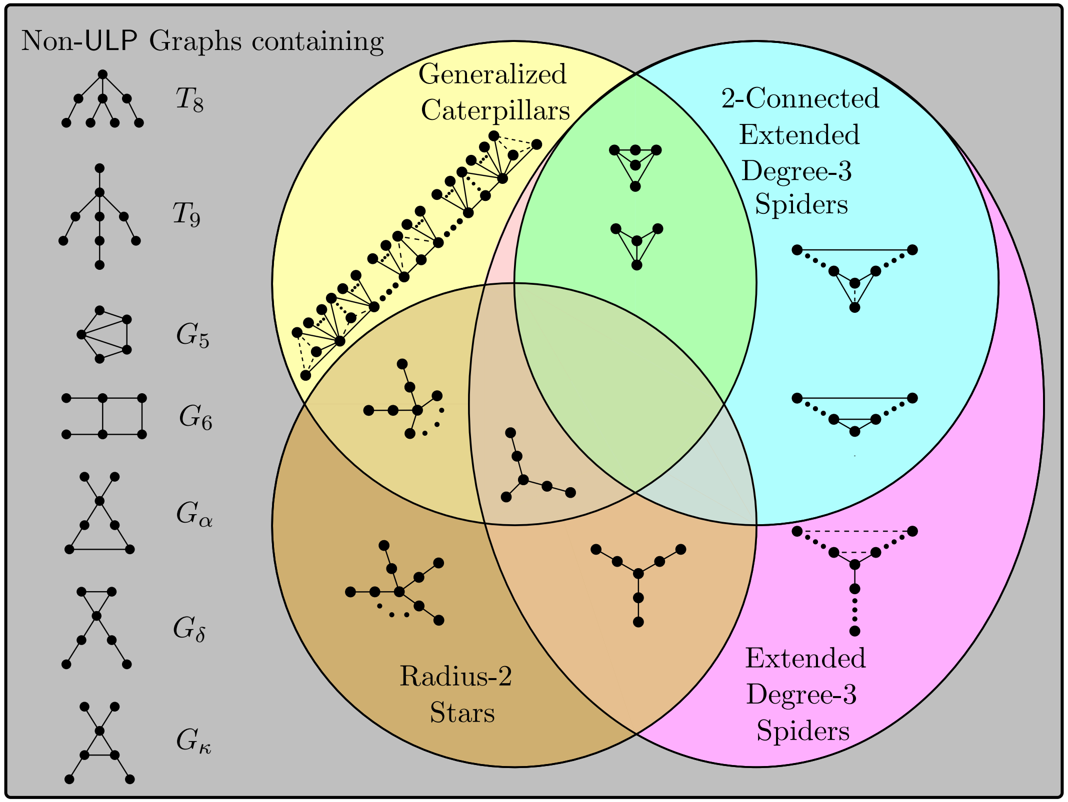
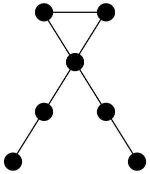


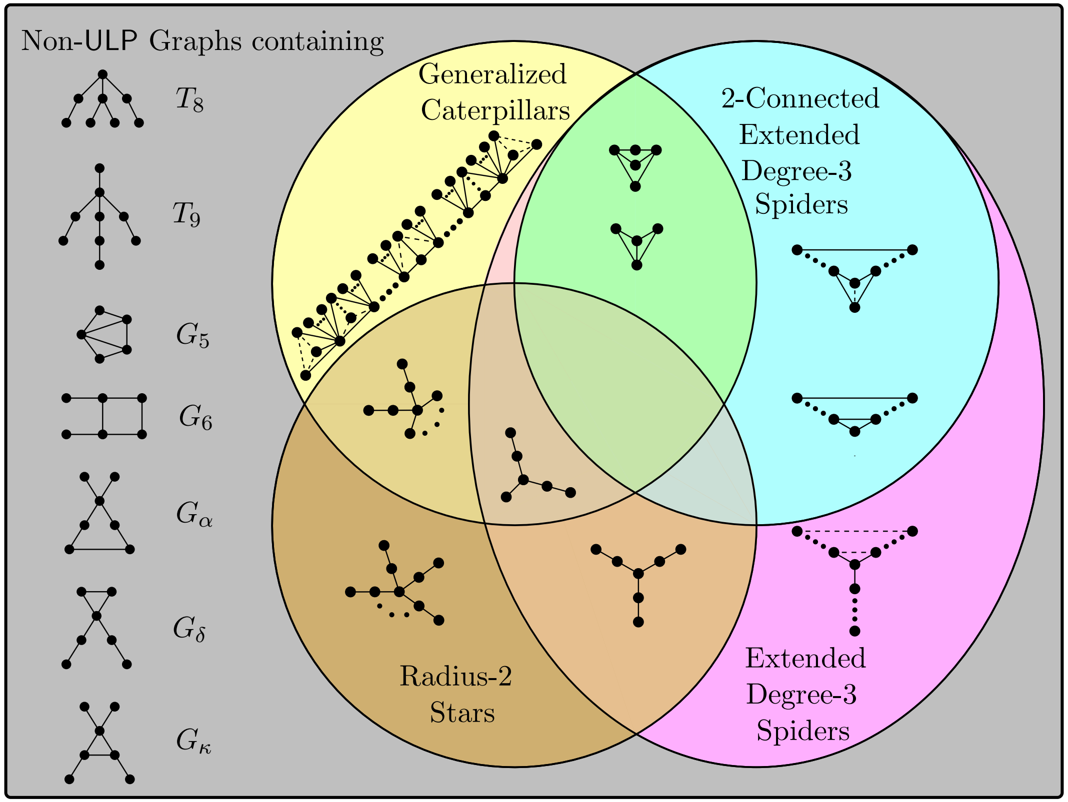
- Characterization of ULP graphs by 7 forbidden subdivisions
  - Graph G_α with 7 vertices and one node of degree 4



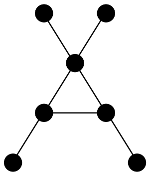


- Characterization of ULP graphs by 7 forbidden subdivisions
  - Graph G_δ with 7 vertices and one node of degree 4





- Characterization of ULP graphs by 7 forbidden subdivisions
  - Graph G_κ with 7 vertices and three nodes of degree 4



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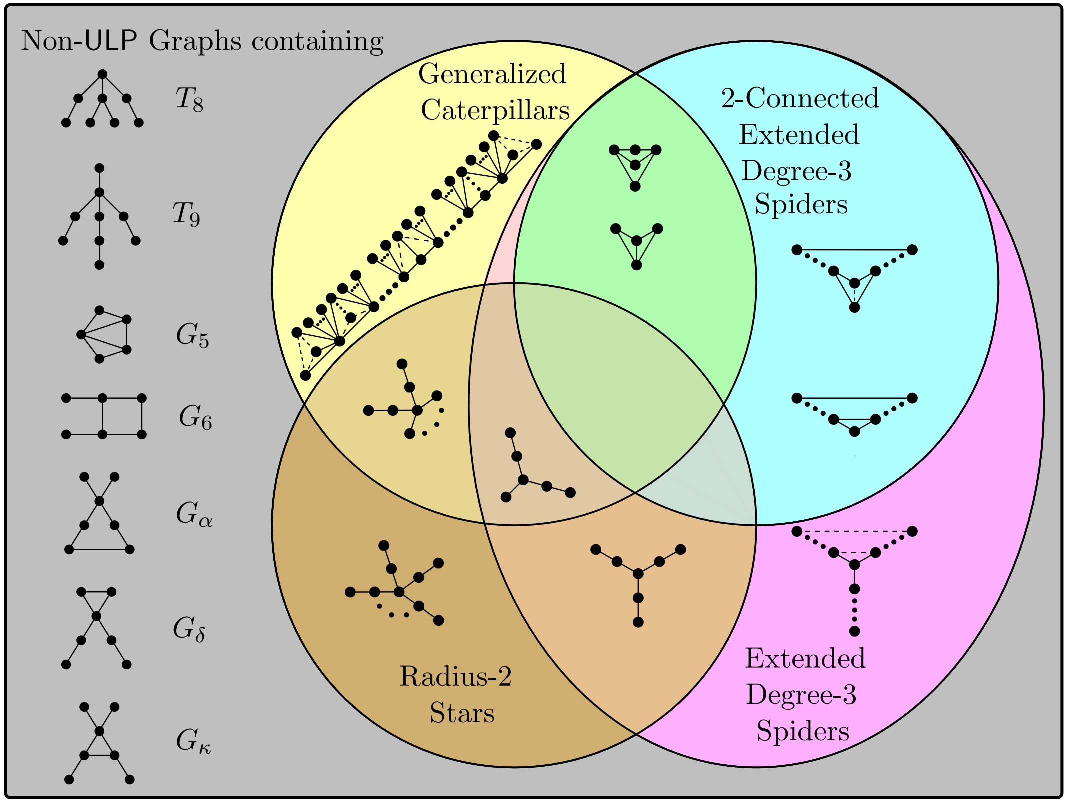
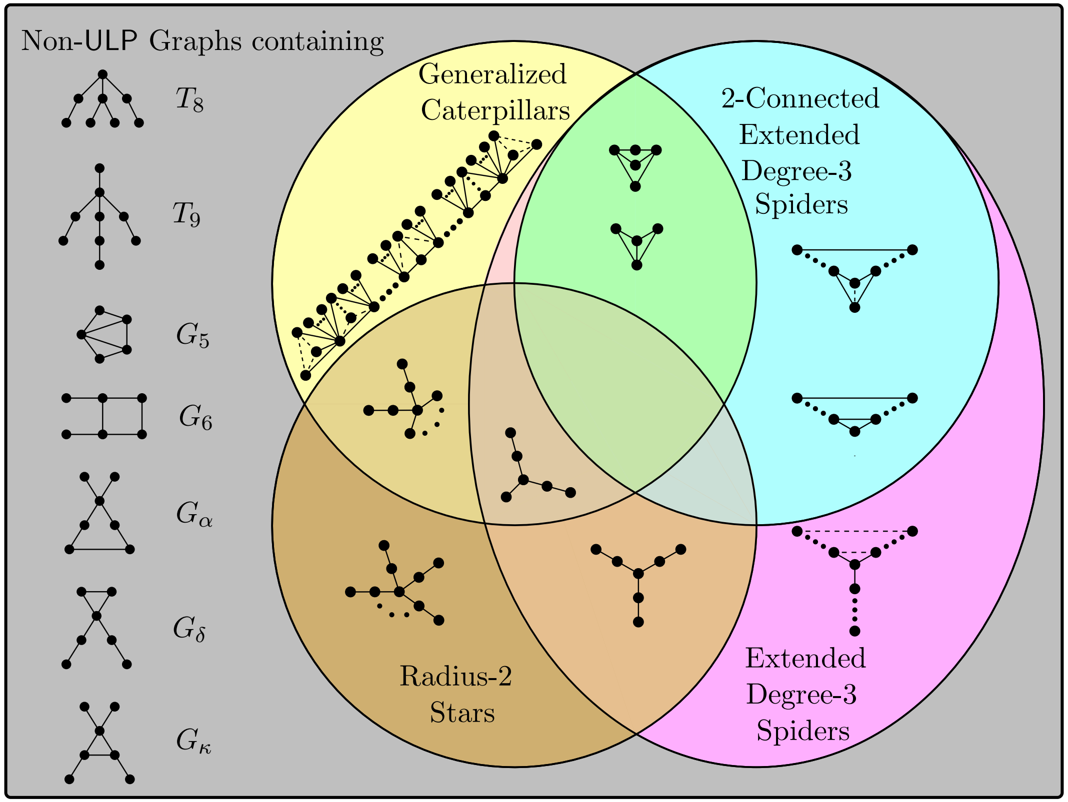
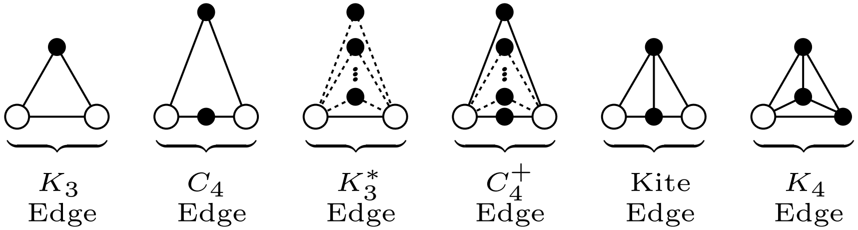


- Characterization of ULP graphs by 7 forbidden subdivisions
- All ULP graphs fall into one of three categories:

New Results – ULP Graphs



- Characterization of ULP graphs by 7 forbidden subdivisions
- All ULP graphs fall into one of three categories:
  - **Generalized Caterpillars**



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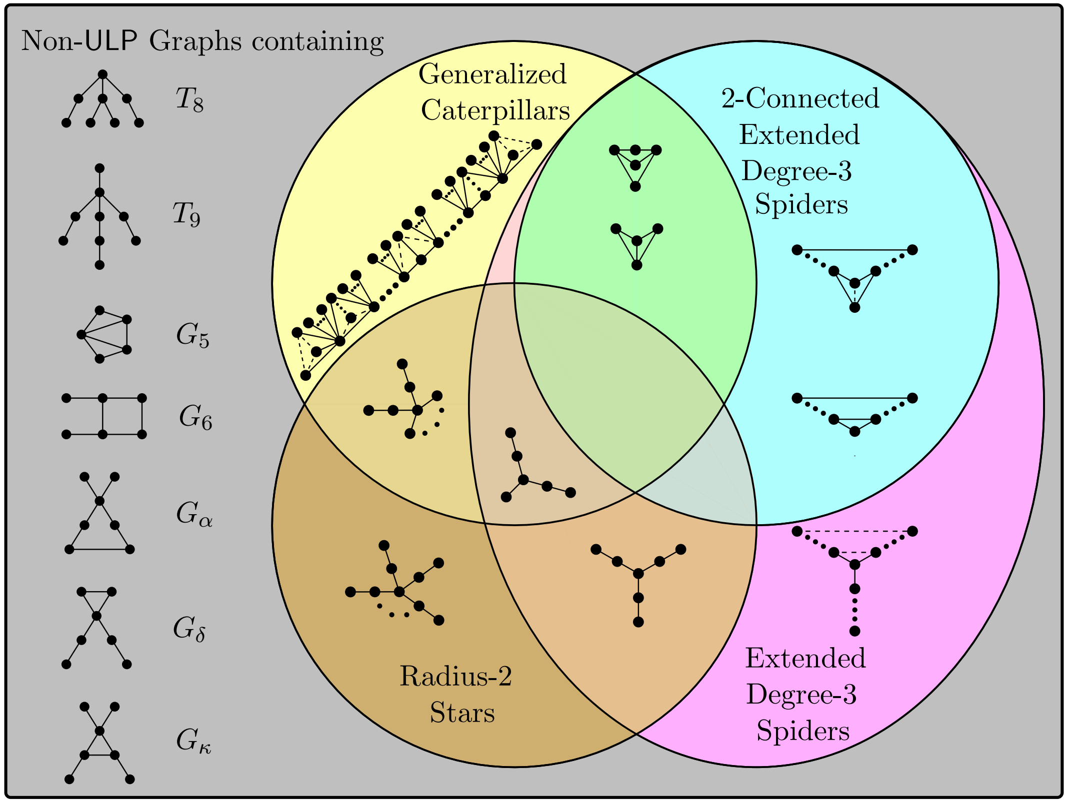

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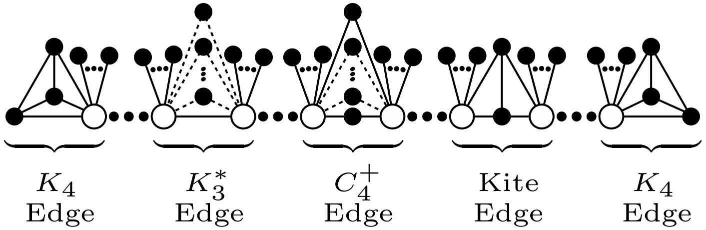

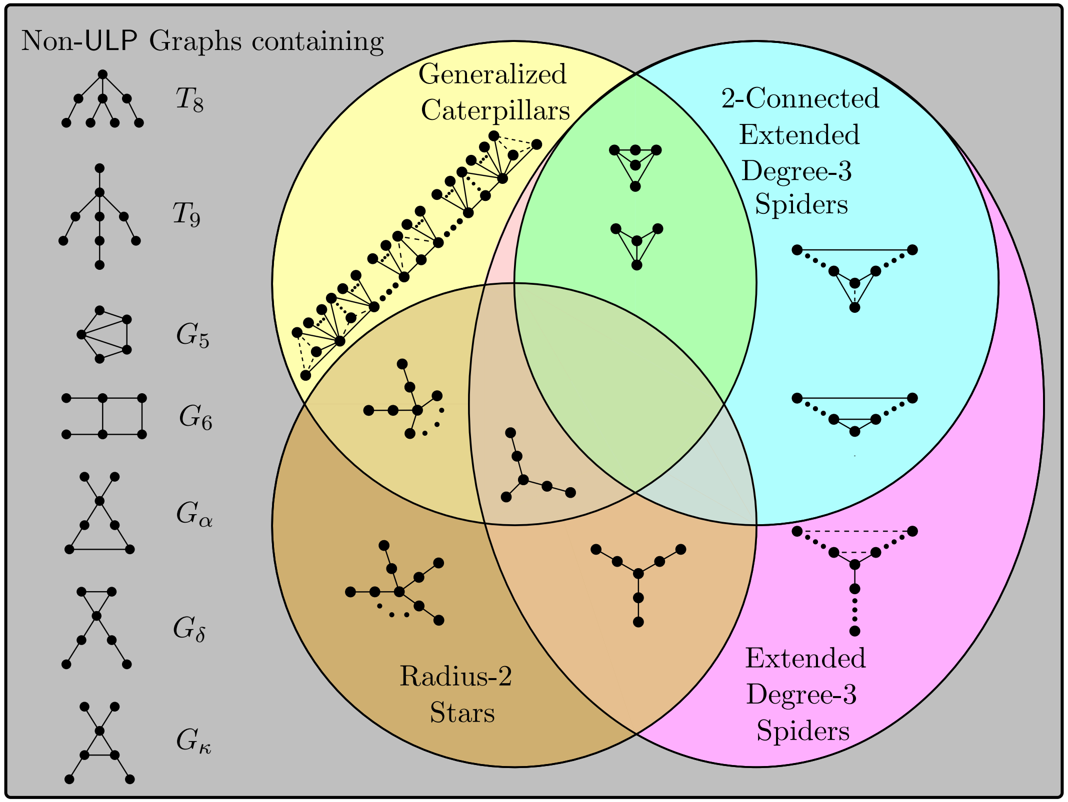

- All ULP graphs fall into one of three categories:
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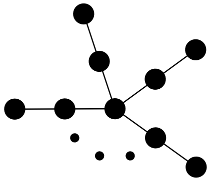
  - **Radius-2 Stars**



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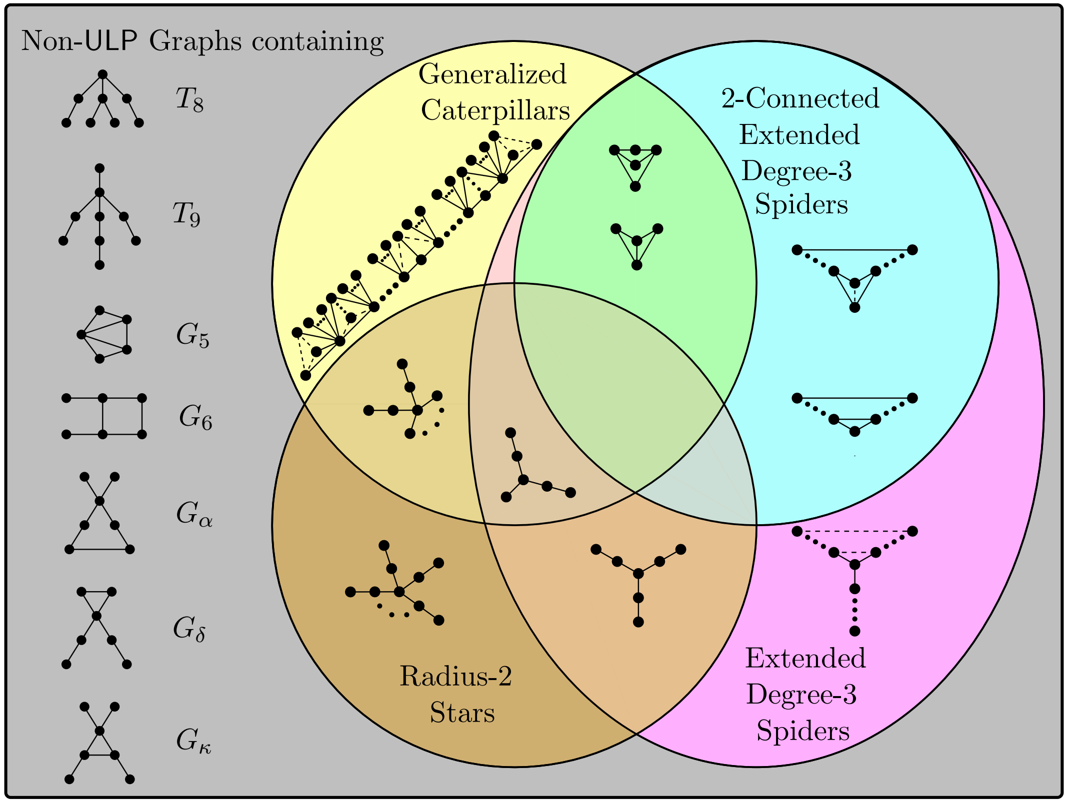

- All ULP graphs fall into one of three categories:
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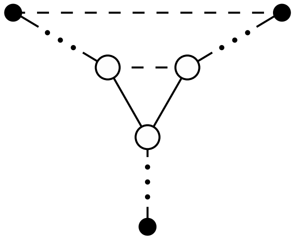
  - **Extended Degree-3 Spiders**



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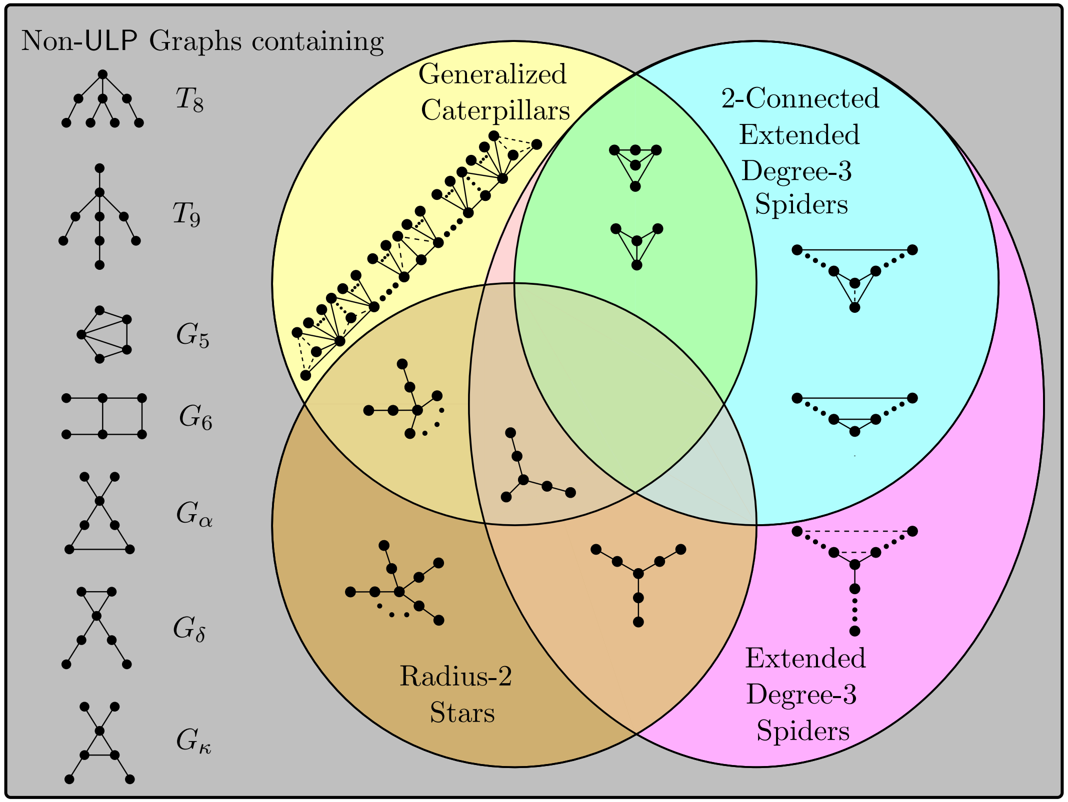

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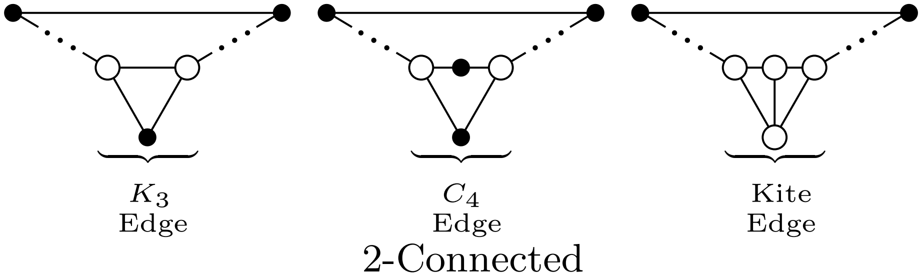
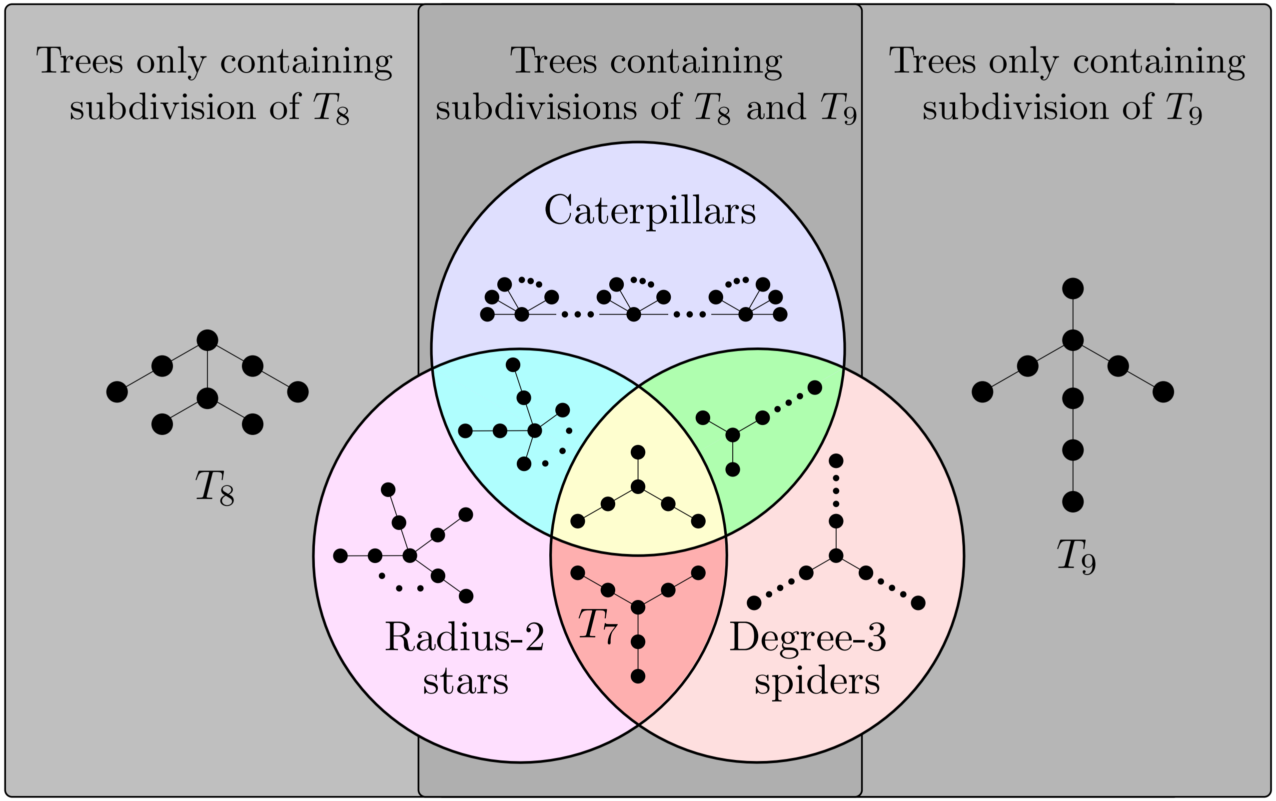


Outline – Unlabeled Level Planar Trees

- Background
- Unlabeled Level Planar Trees



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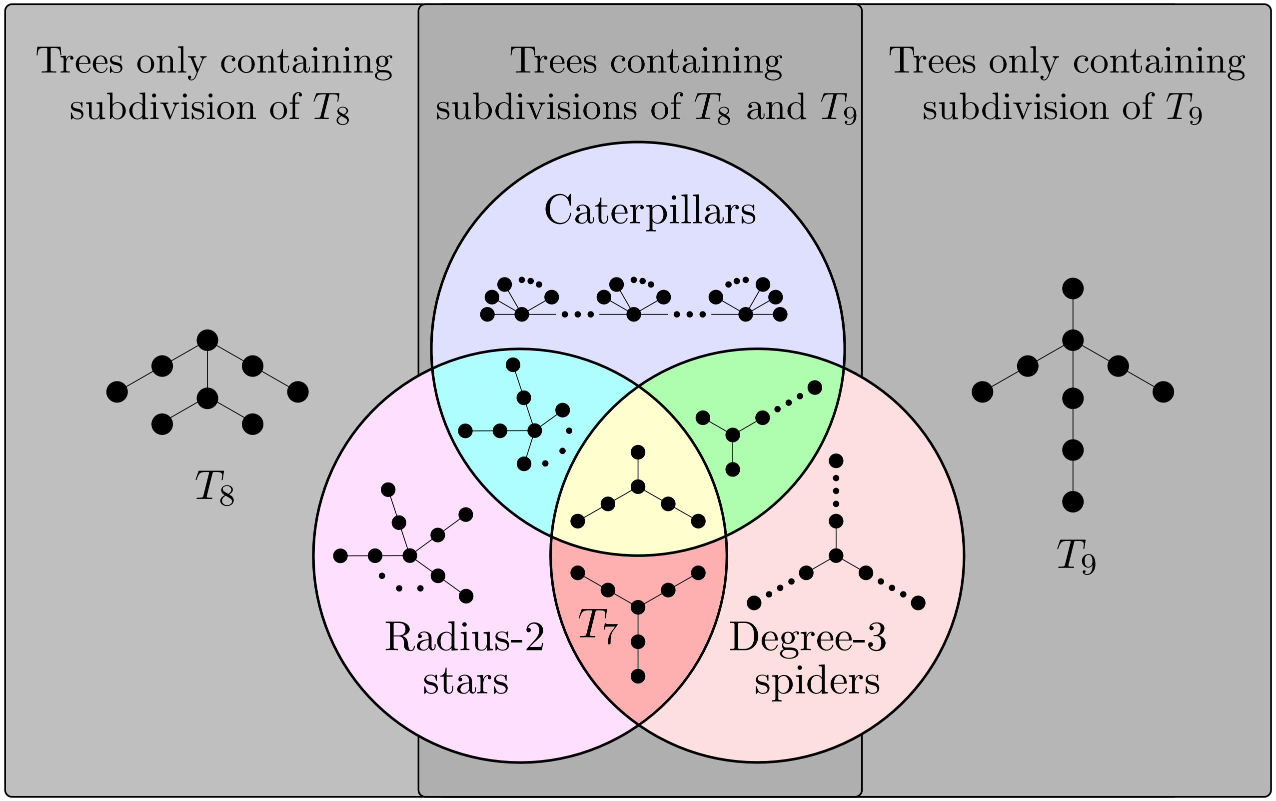

  - First show that forbidden trees are not ULP

Outline – Unlabeled Level Planar Trees

- Background
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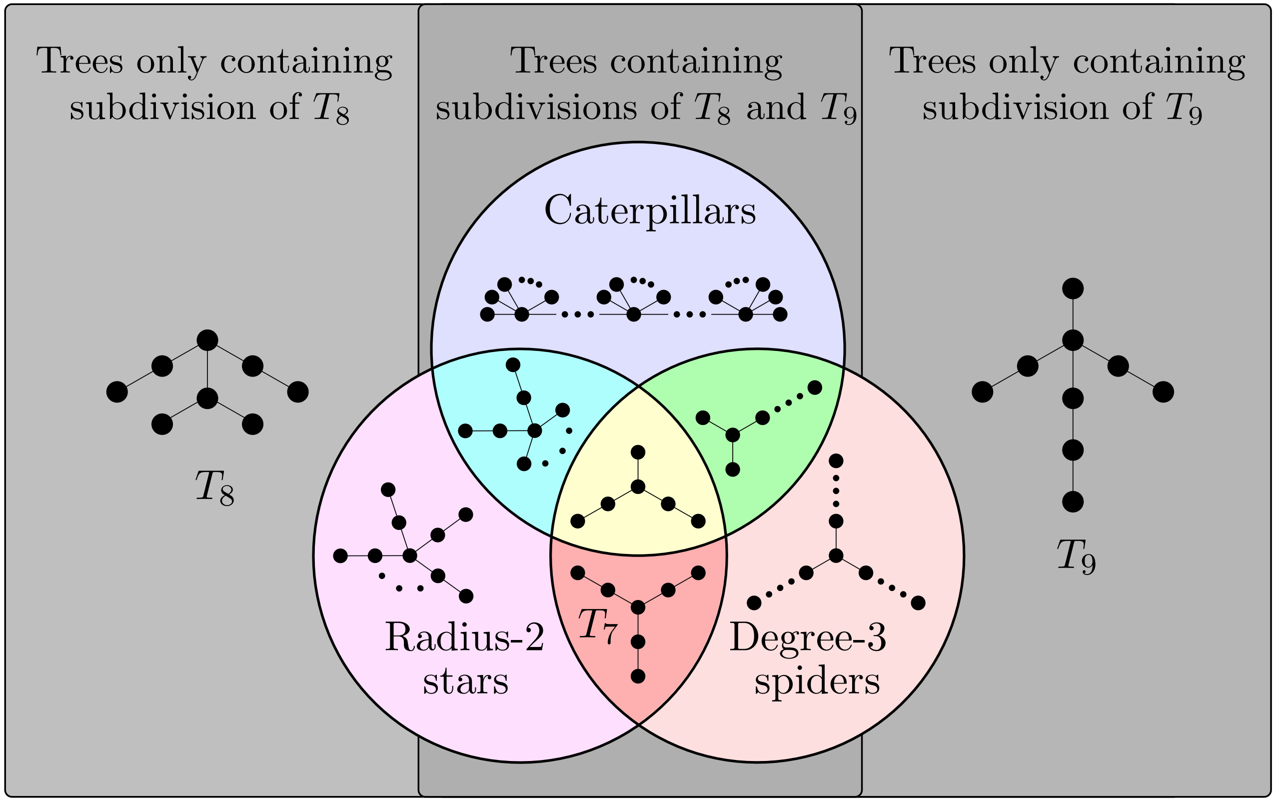

- Then show how to draw ULP trees

Outline – Unlabeled Level Planar Trees

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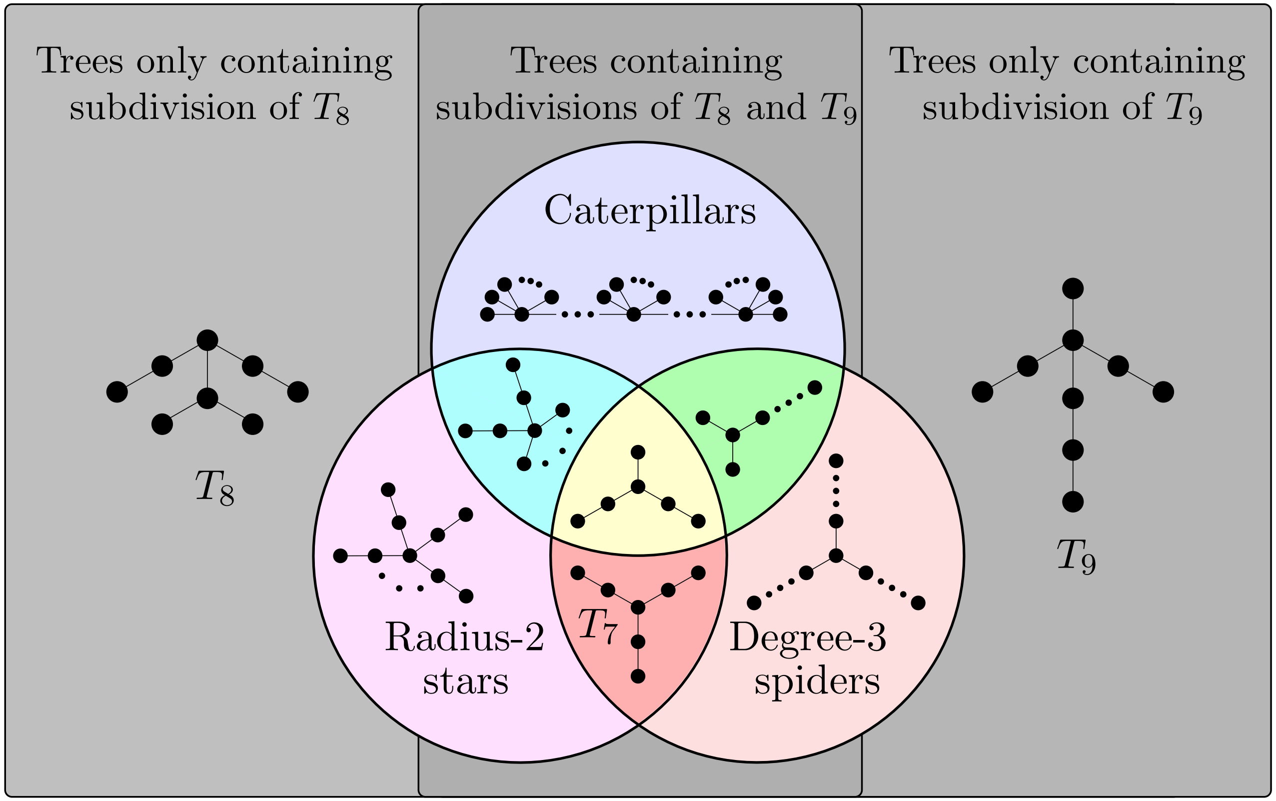

- Finally describe how all trees either
  - Contain a subdivision of T_8 or T_9 OR
  - Are one of the three ULP trees

Key Observation

- Let C be some chain $a{-}b{-}c{-}d{-}e$

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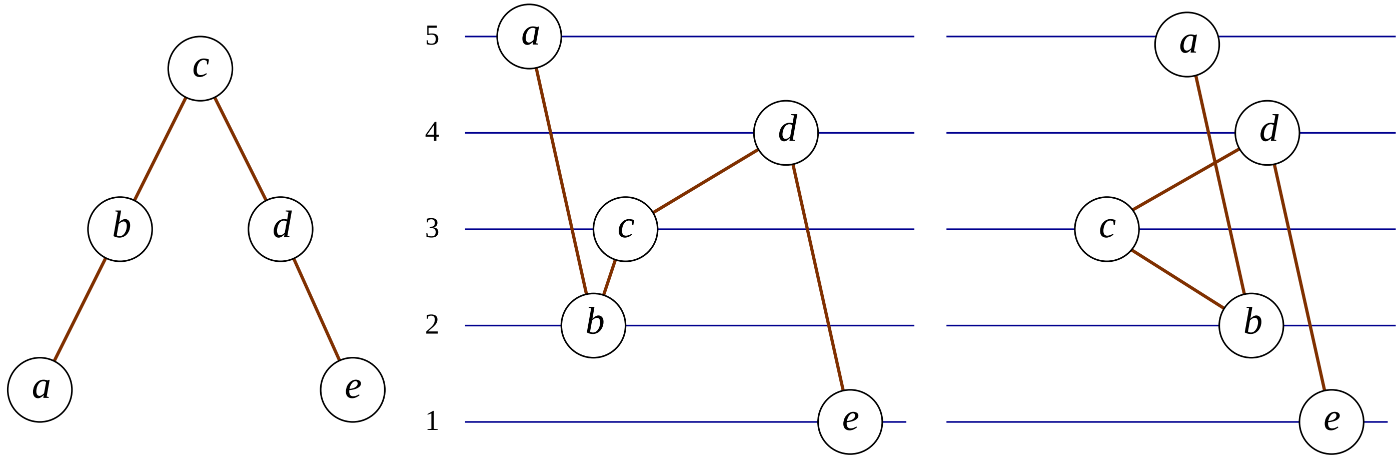

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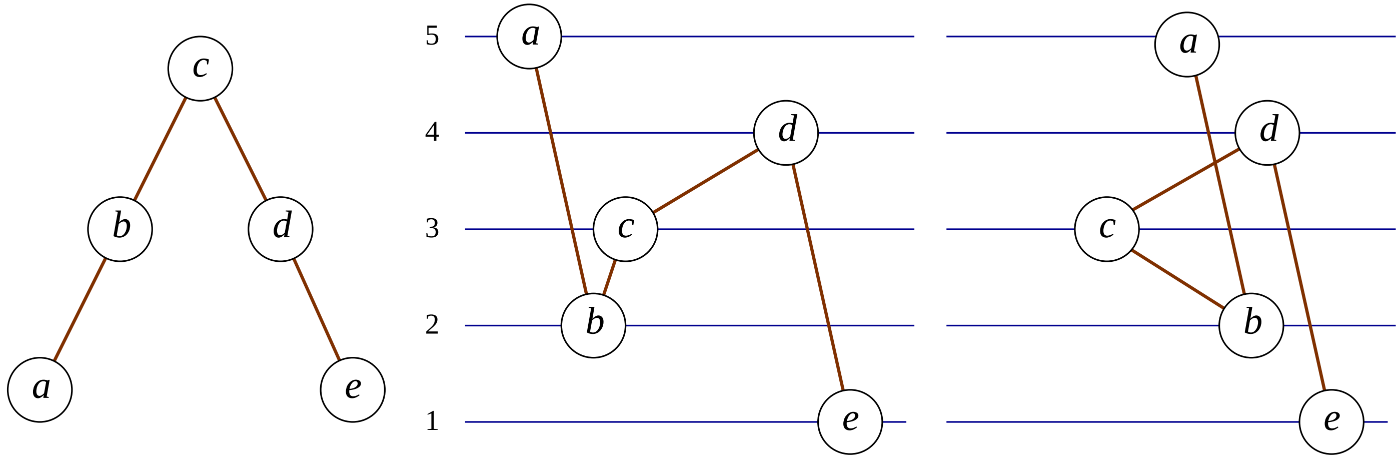

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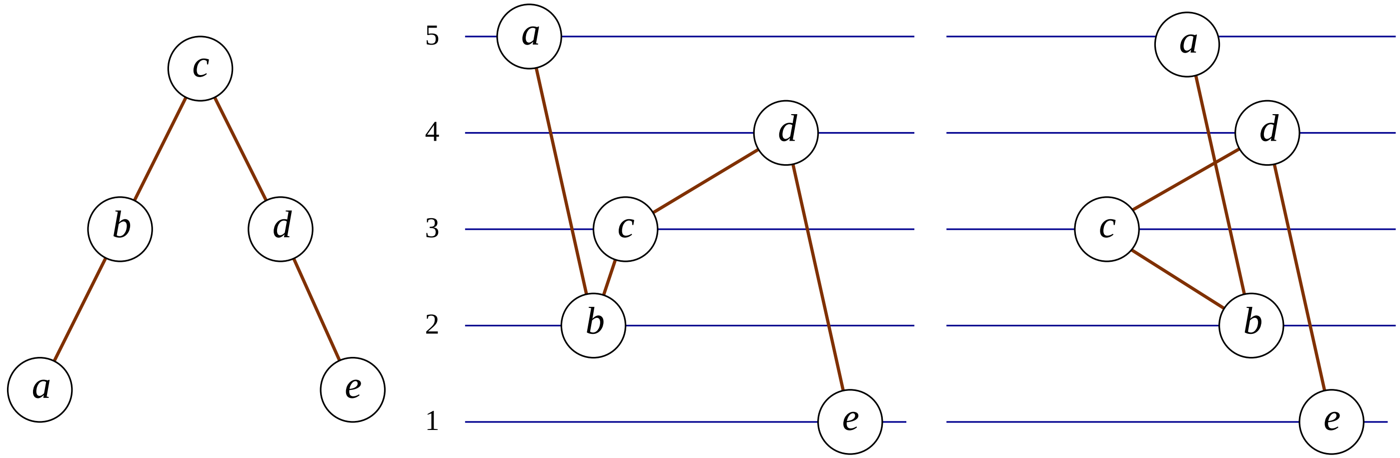
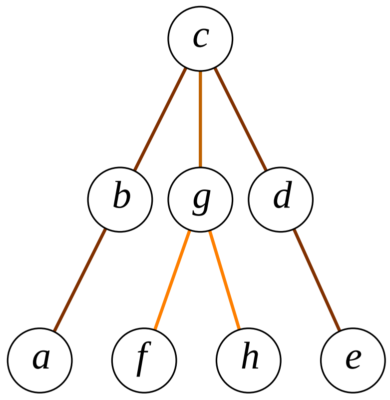


- c cannot be leftmost or rightmost without forcing a crossing
  - Can use this property to prove T_8 and T_9 are not ULP

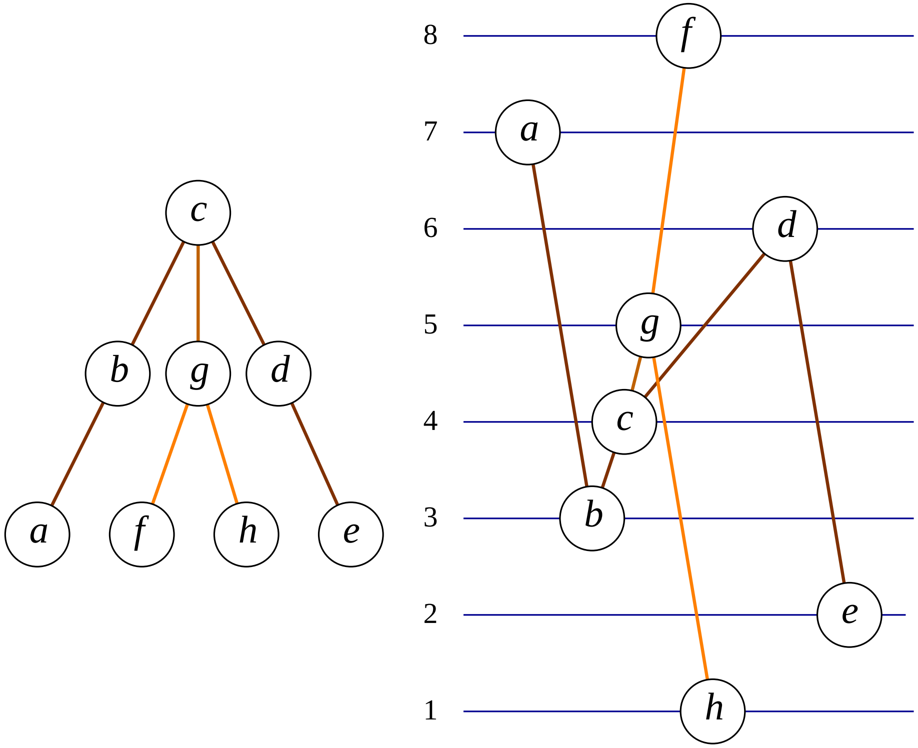
Forbidden Trees – T_8

- Let C be the chain a–b–c–d–e



Forbidden Trees – T_8

- Let C be the chain $a{-}b{-}c{-}d{-}e$
  - $\phi(\{a, f\}) < \phi(d) < \phi(\{c, g\}) < \phi(b) < \phi(\{e, h\})$



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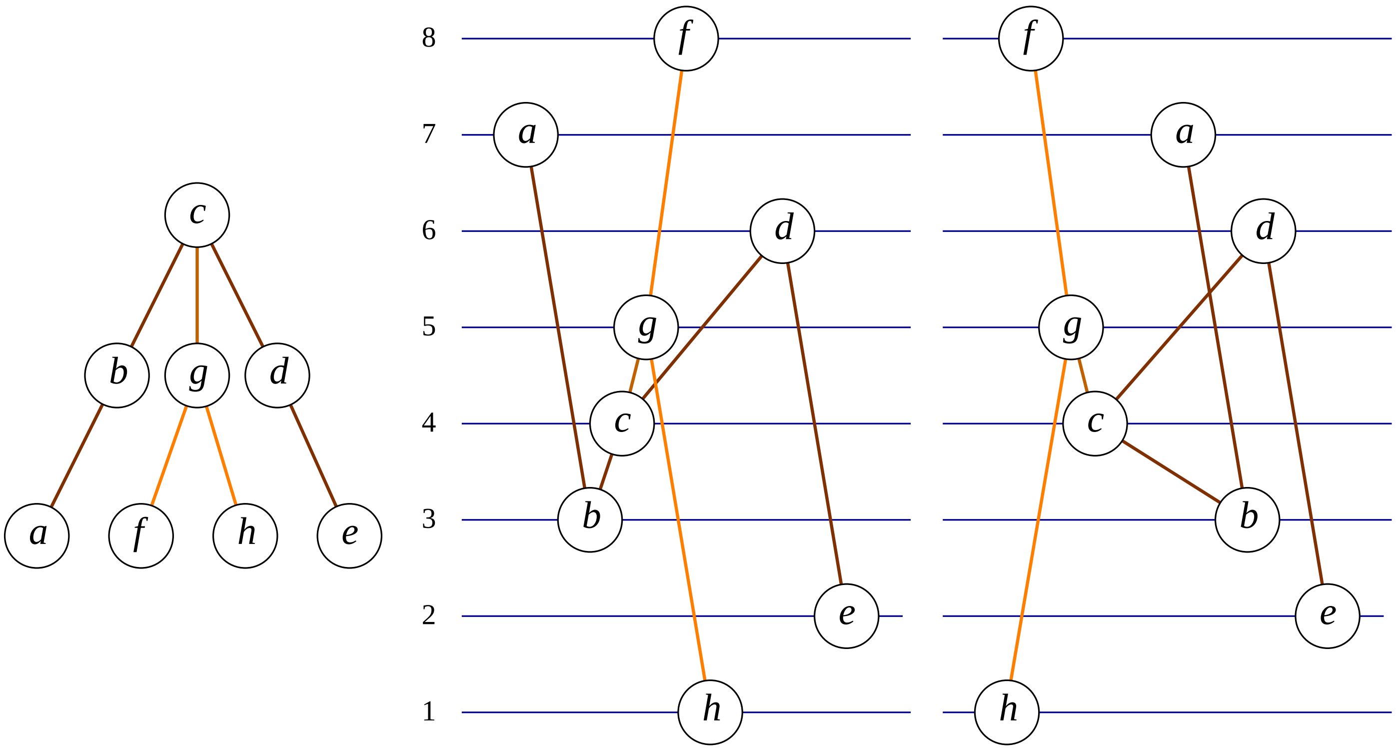

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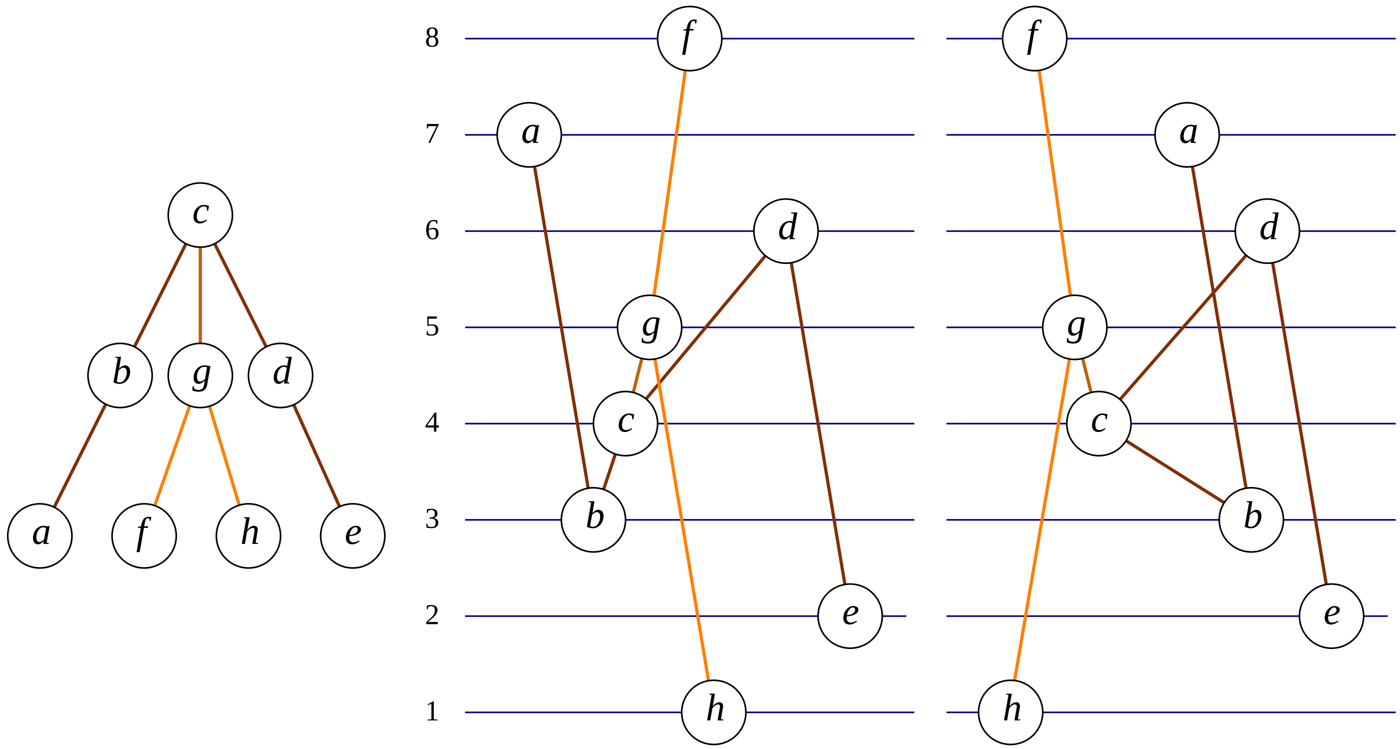
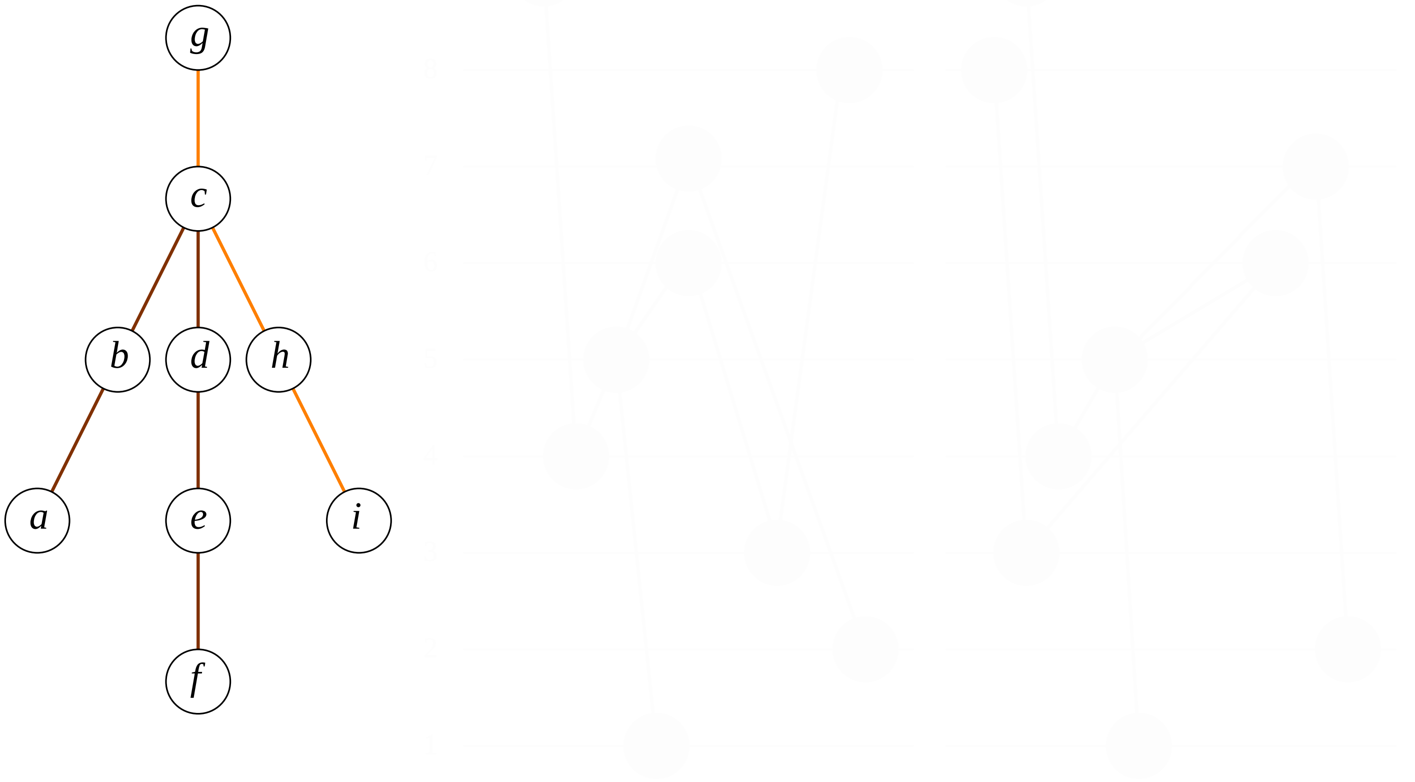


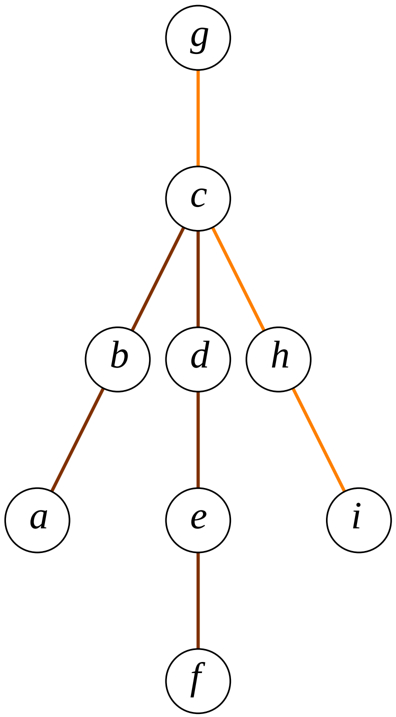
Lemma 1 *Any tree containing a subdivision of* T_8 *cannot be ULP.*

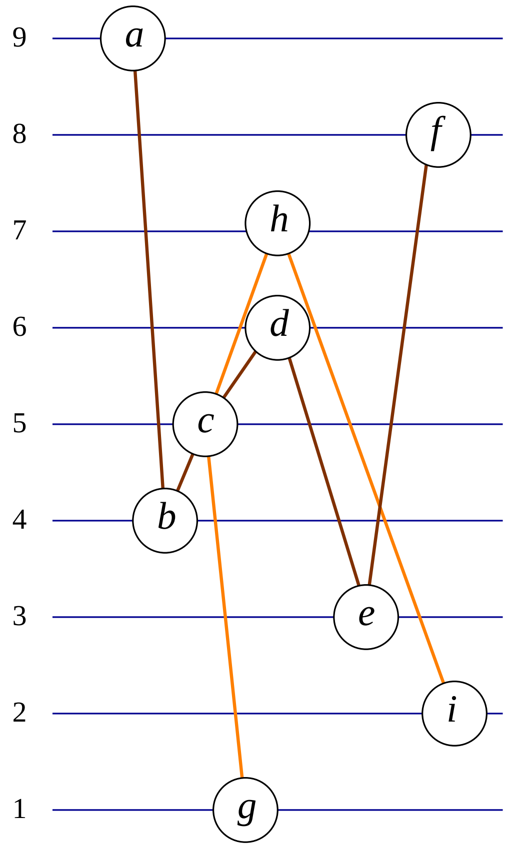
- Let C be the chain $a{-}b{-}c{-}d{-}e$



Forbidden Trees – T_9

- Let C be the chain $a{-}b{-}c{-}d{-}e$
  - $\phi(\{a, f\}) < \phi(h) < \phi(d) < \phi(c) < \phi(b) < \phi(e) < \phi(\{g, i\})$





Forbidden Trees – T_9

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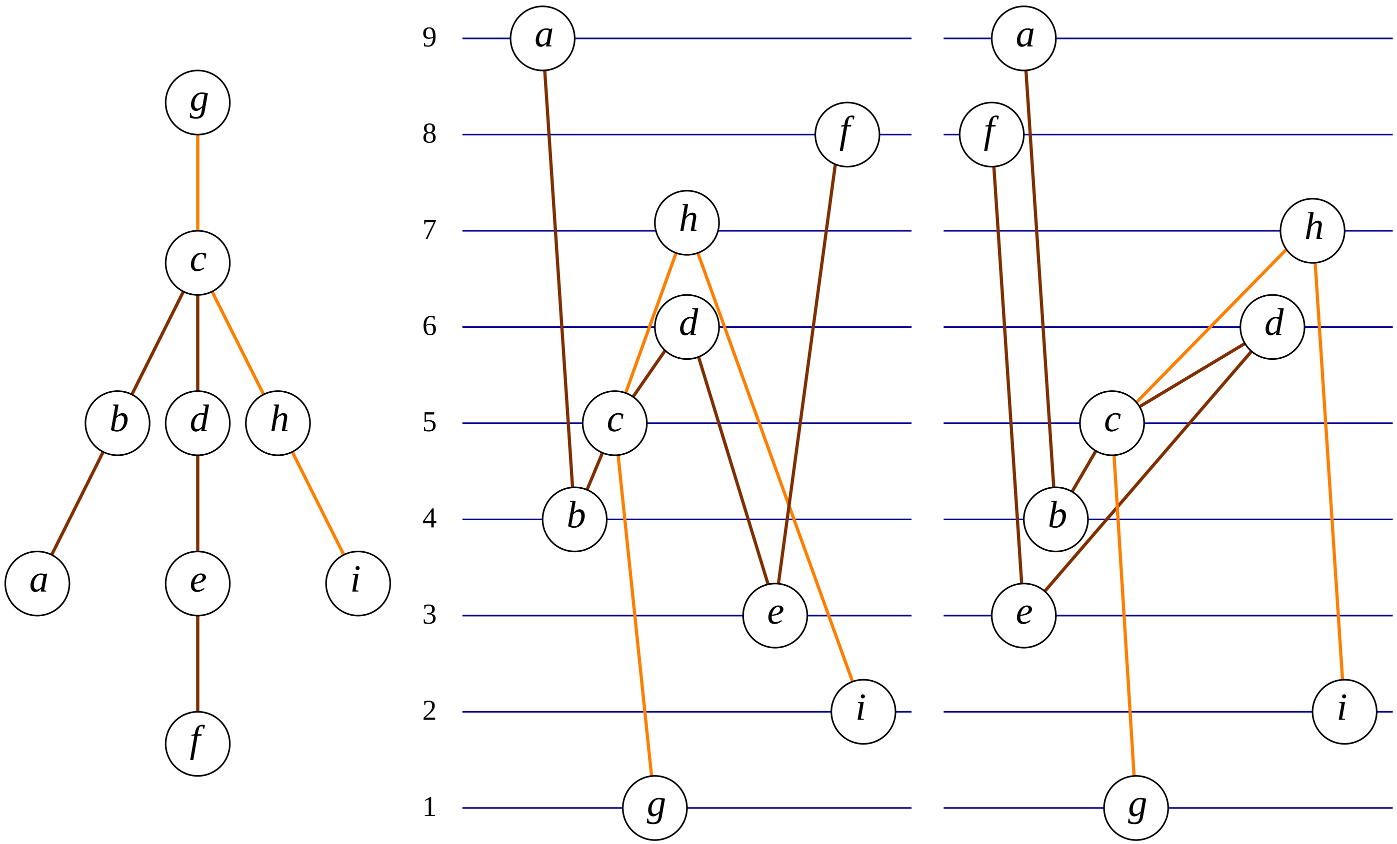

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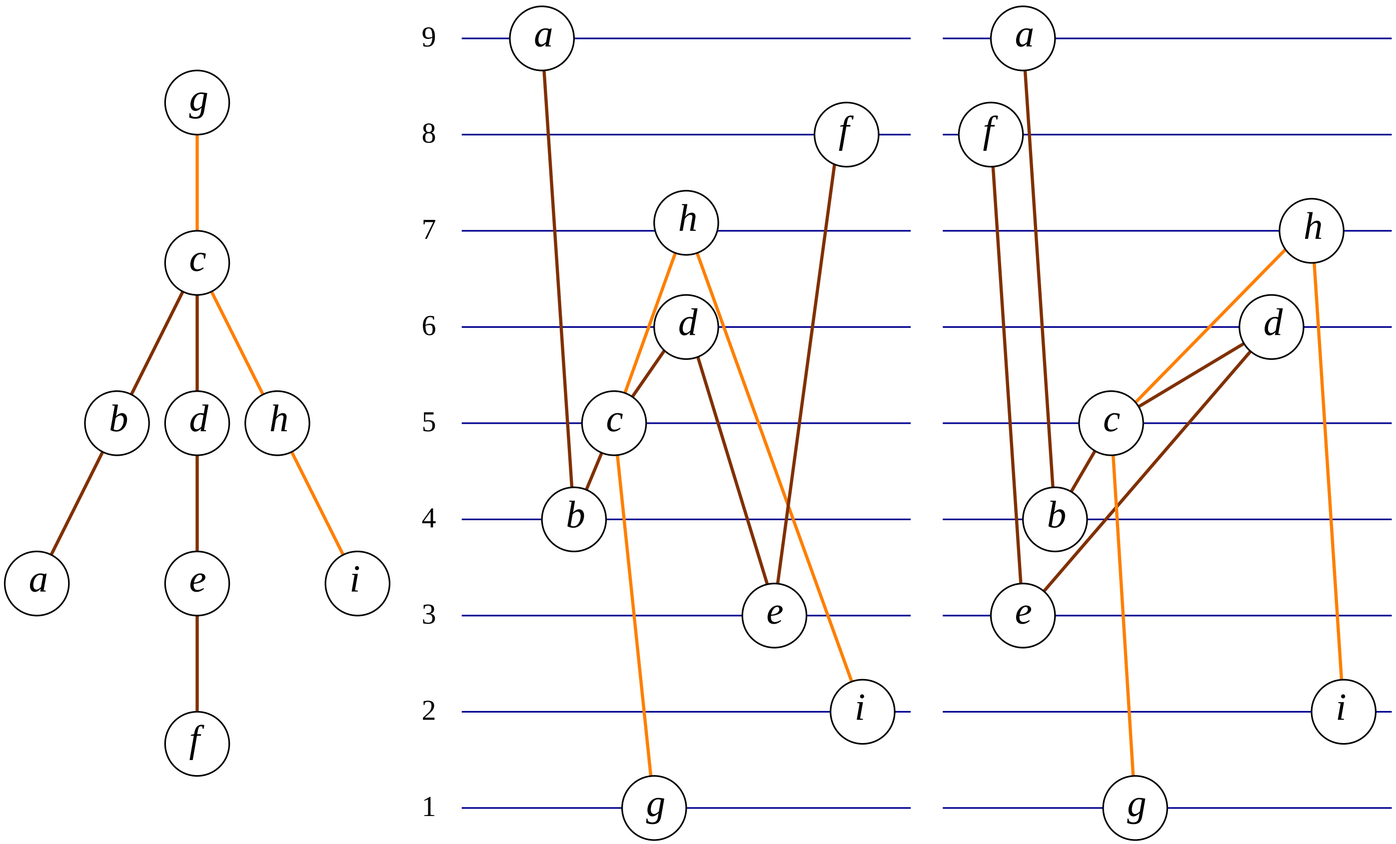
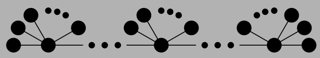


Lemma 2 *Any tree containing T_9 cannot be ULP.*

Drawing Algorithms – Caterpillar 

- Drawing a caterpillar in linear time:

Drawing Algorithms – Caterpillar

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Lemma 3 *(Brass* et al.*, 2003) A plane drawing of n-vertex caterpillar $T(V, E)$ with an m-vertex spine can be drawn in $O(n)$ time on a $2m \times n$ grid for any leveling $\phi : V \xrightarrow[\text{onto}]{1:1} \{1, 2, \ldots, n\}$.*

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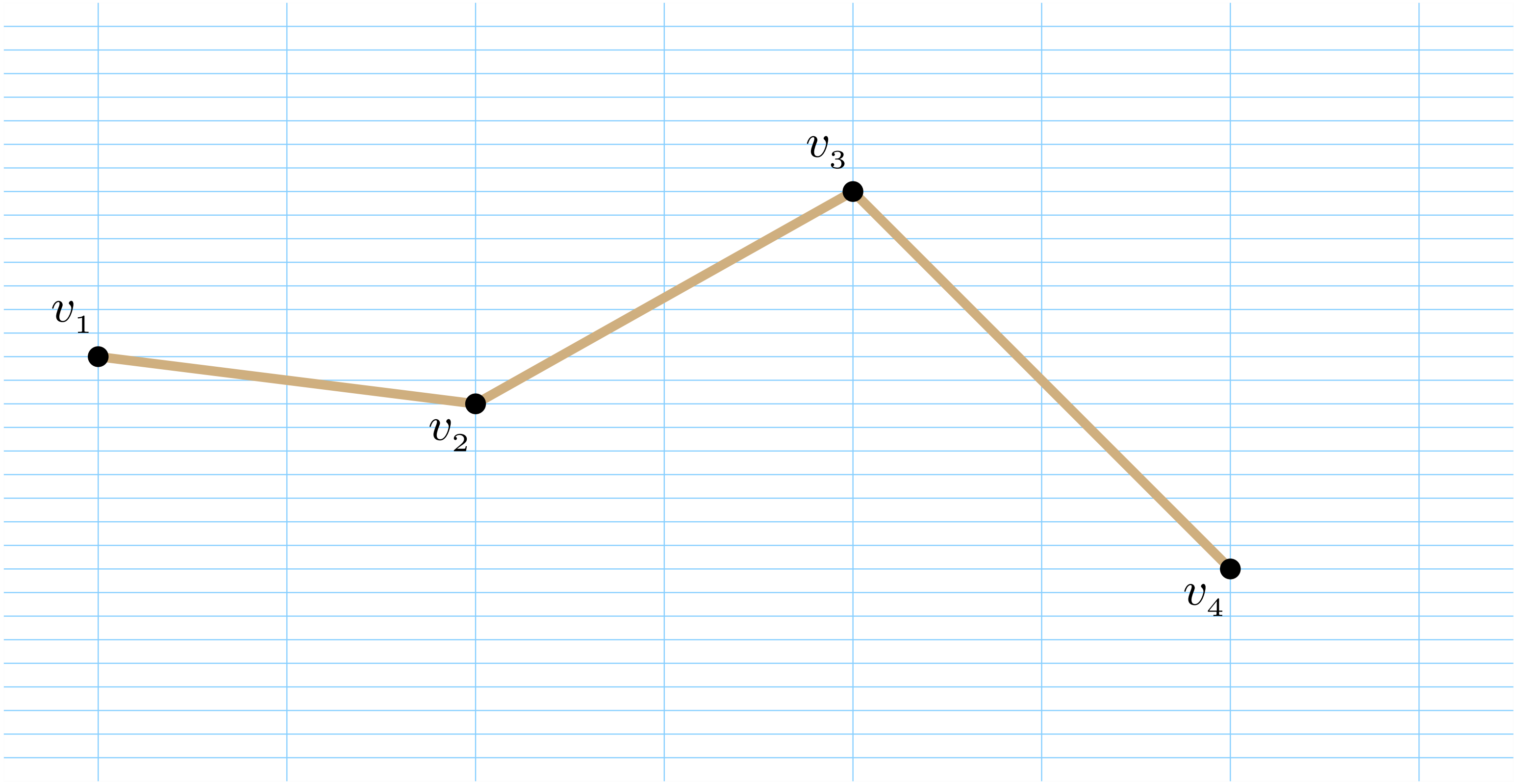

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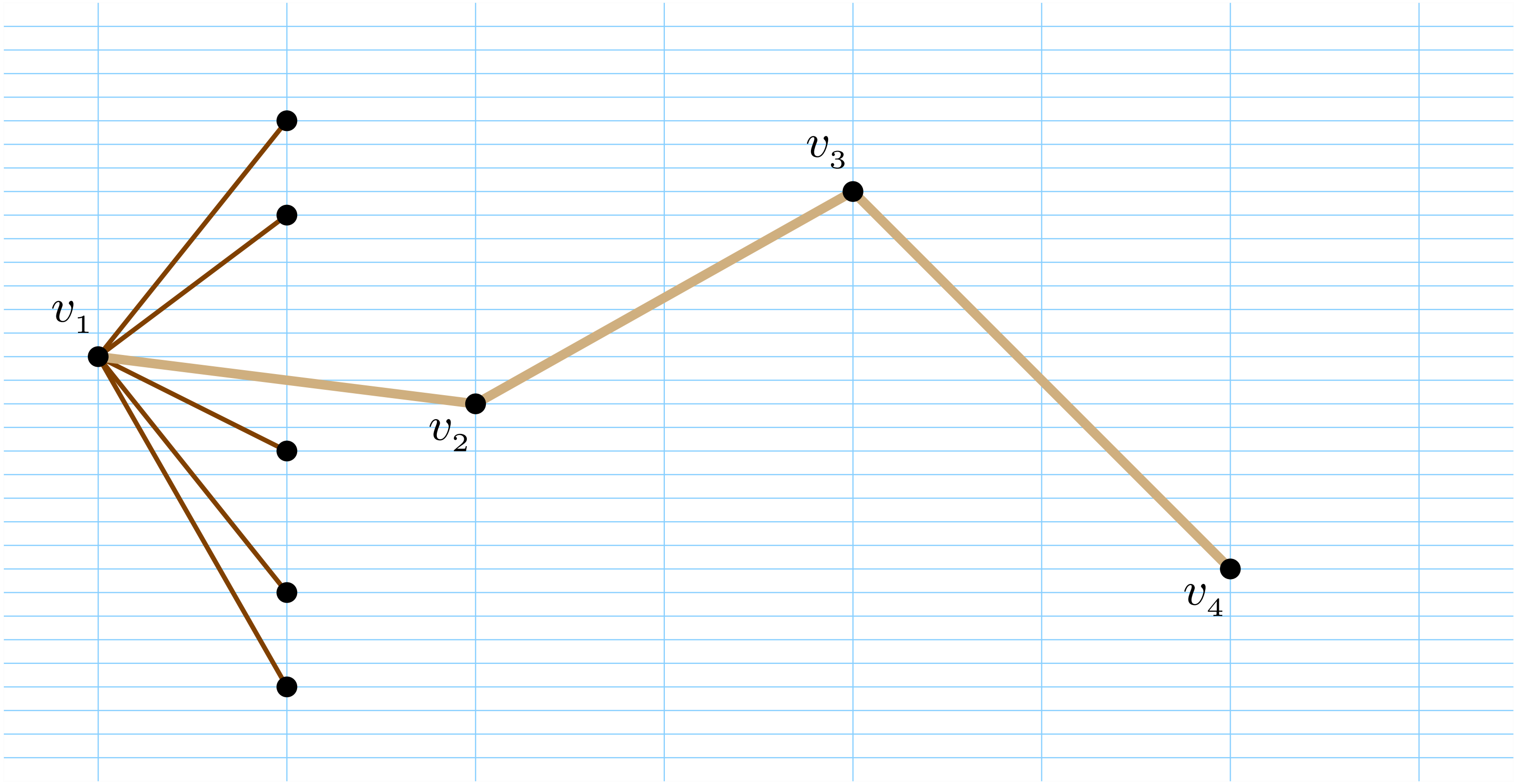

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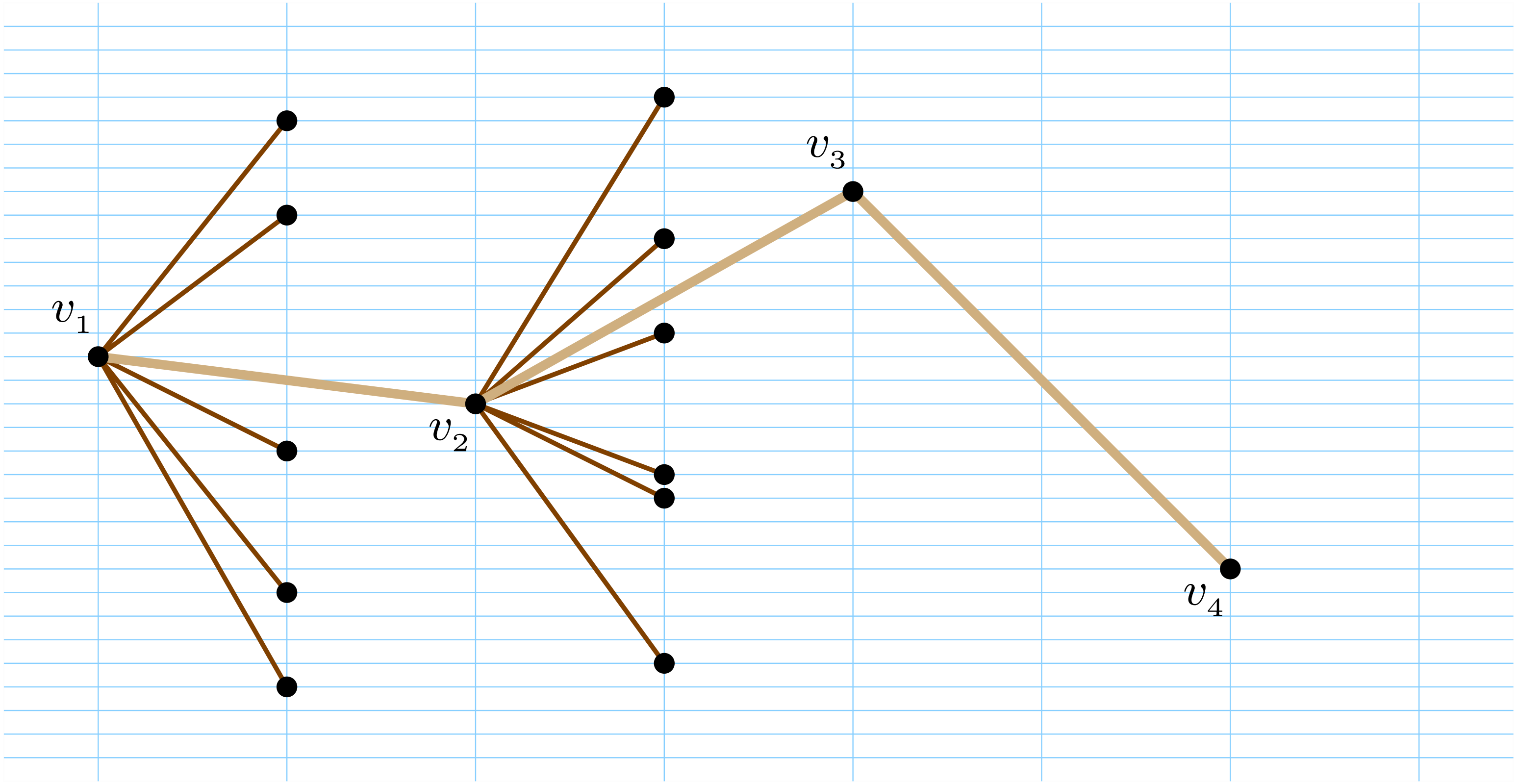

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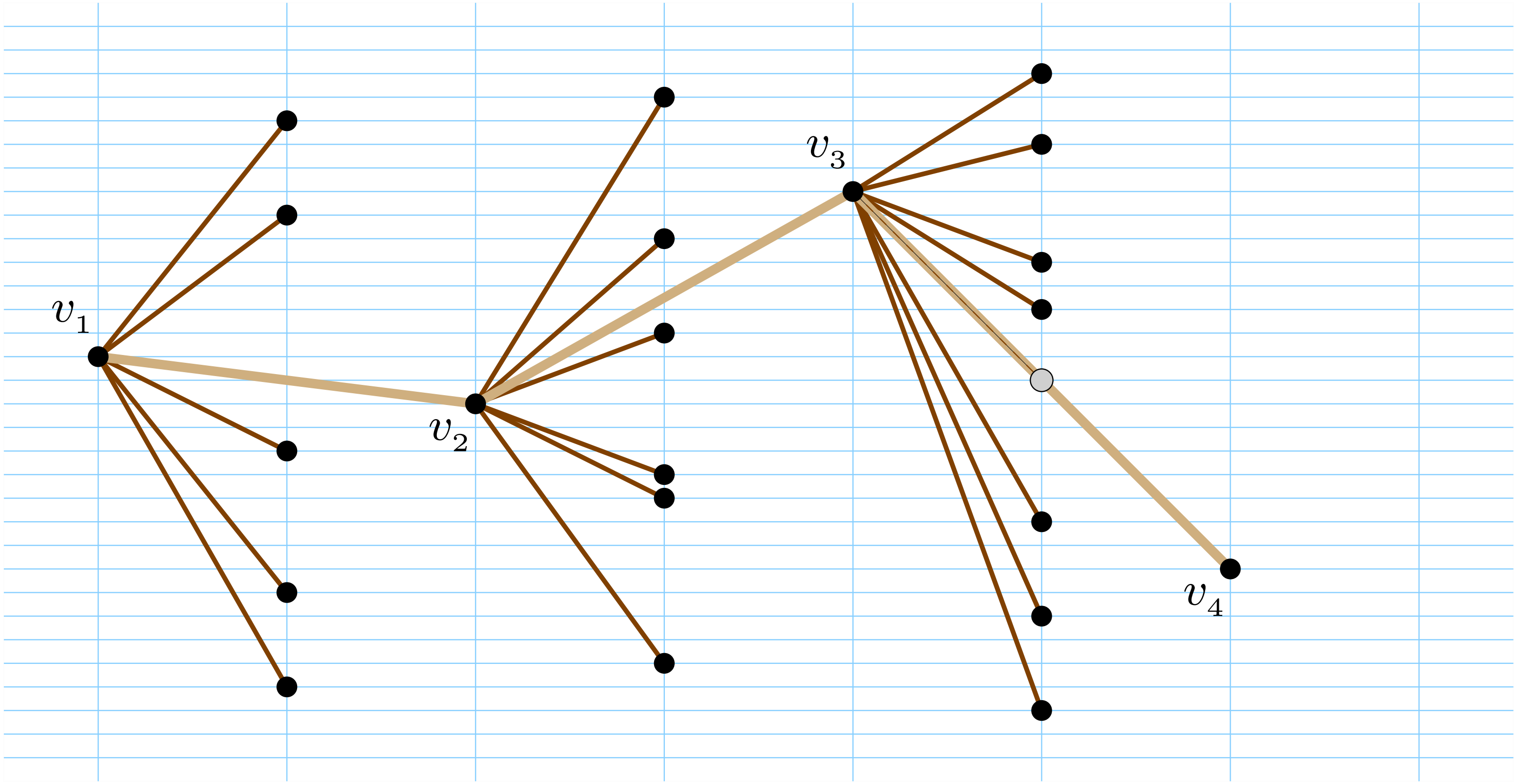

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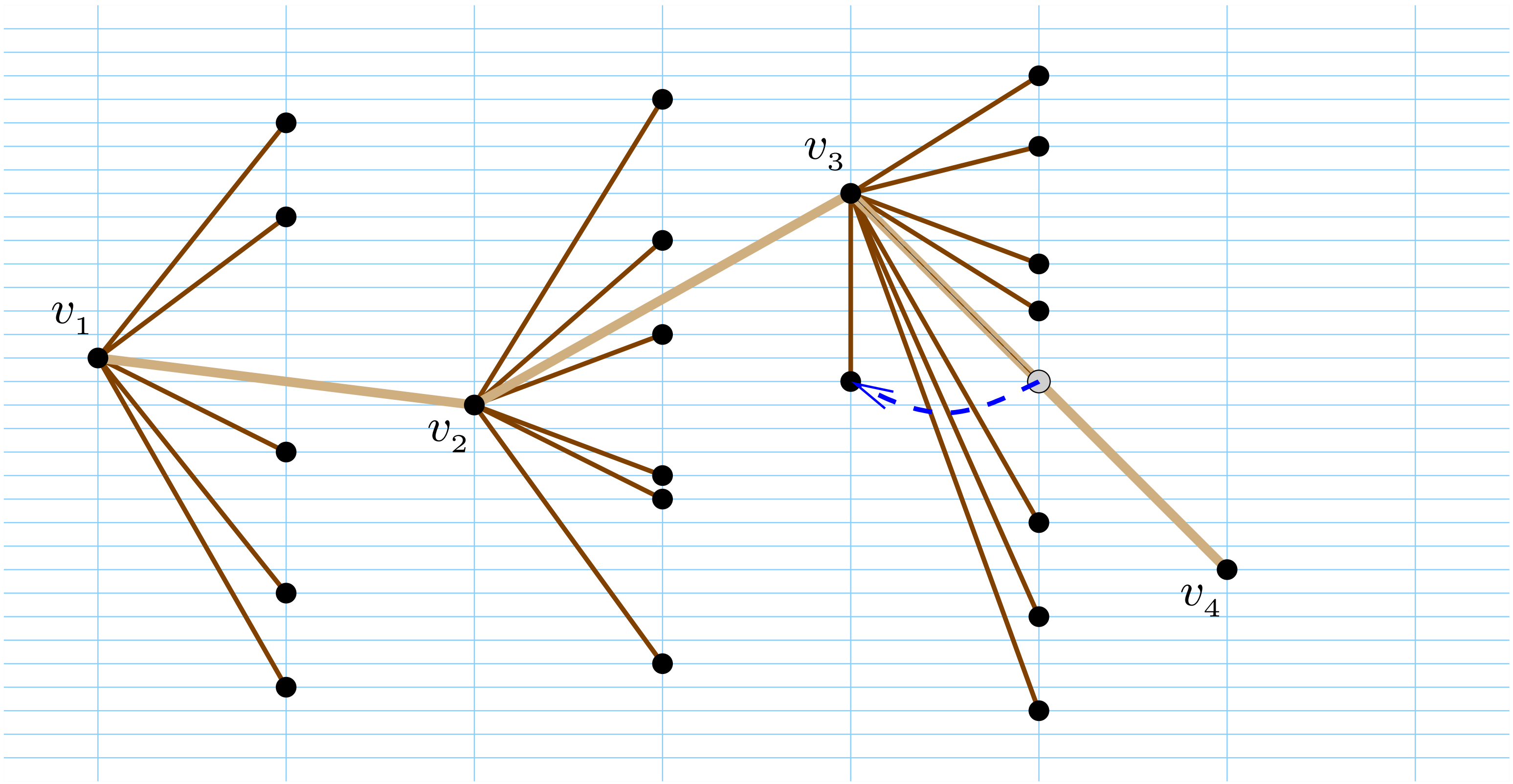

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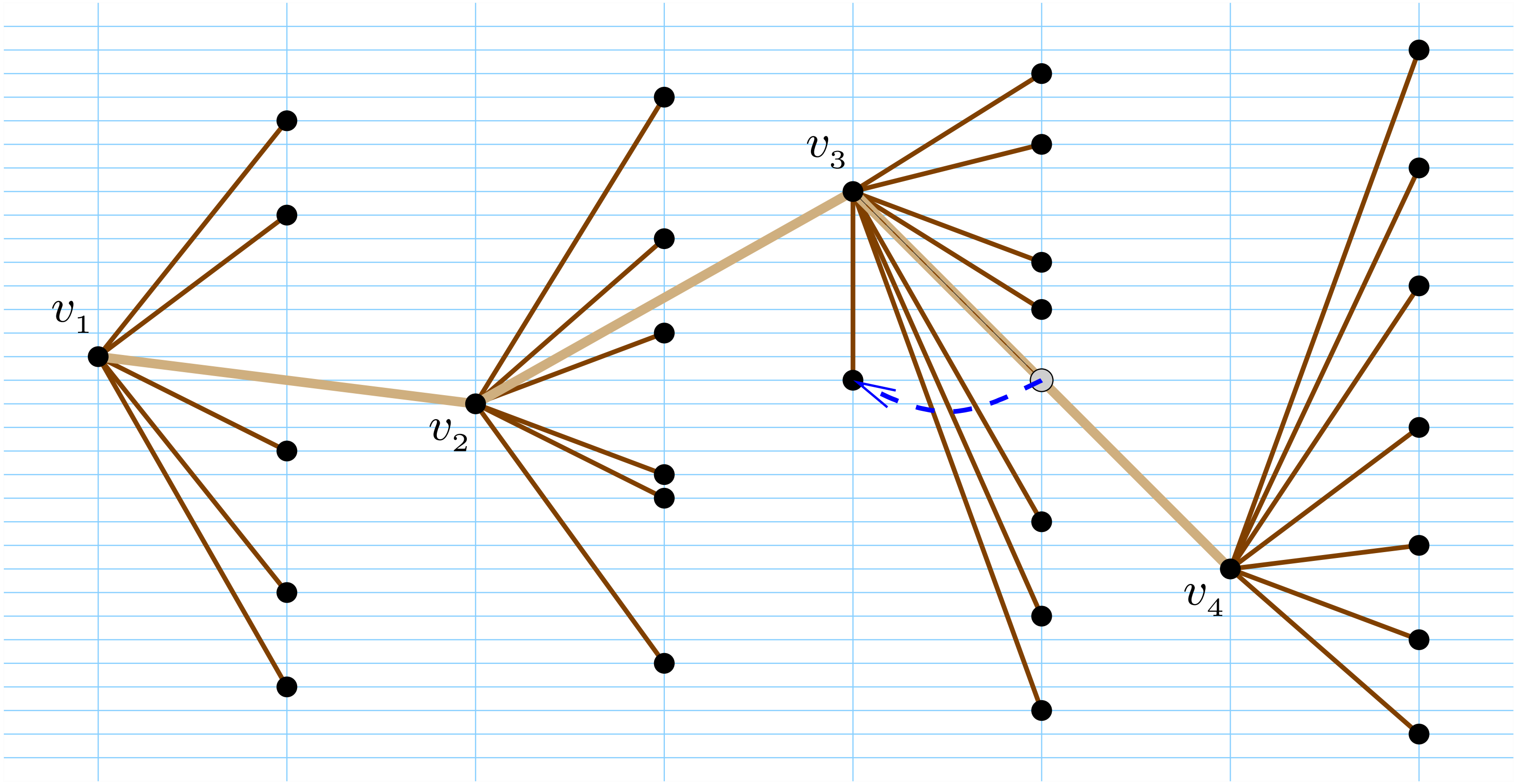

Drawing Algorithms – Radius-2 Stars

- Drawing a radius-2 star in linear time:

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Lemma 4 *A plane drawing of a n-vertex radius-2 star $T(V, E)$ can be drawn in $O(n)$ time on a $(2n+1) \times n$ grid for any leveling $\phi : V \xrightarrow[\text{onto}]{1:1} \{1, 2, \ldots, n\}$.*

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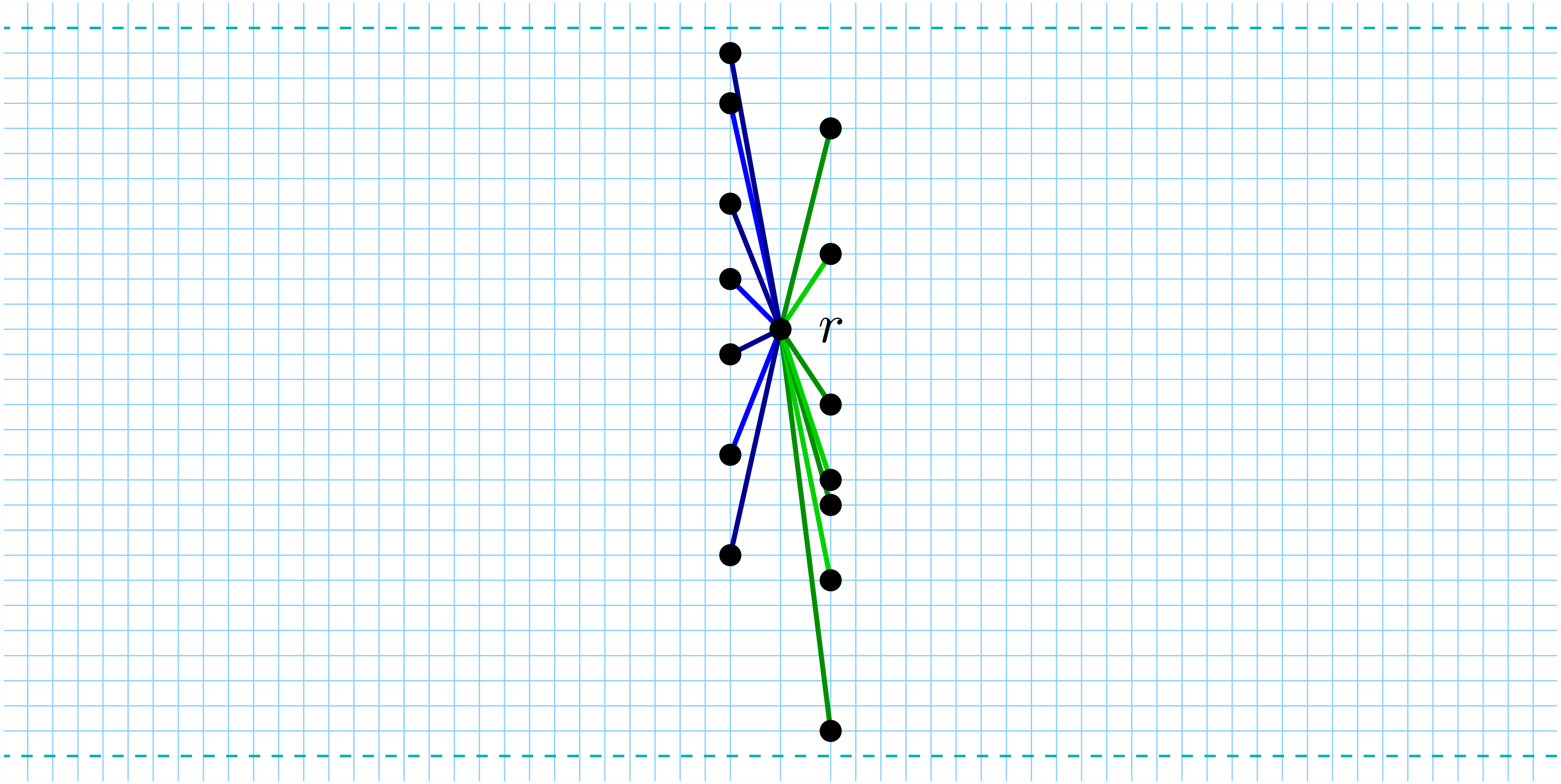

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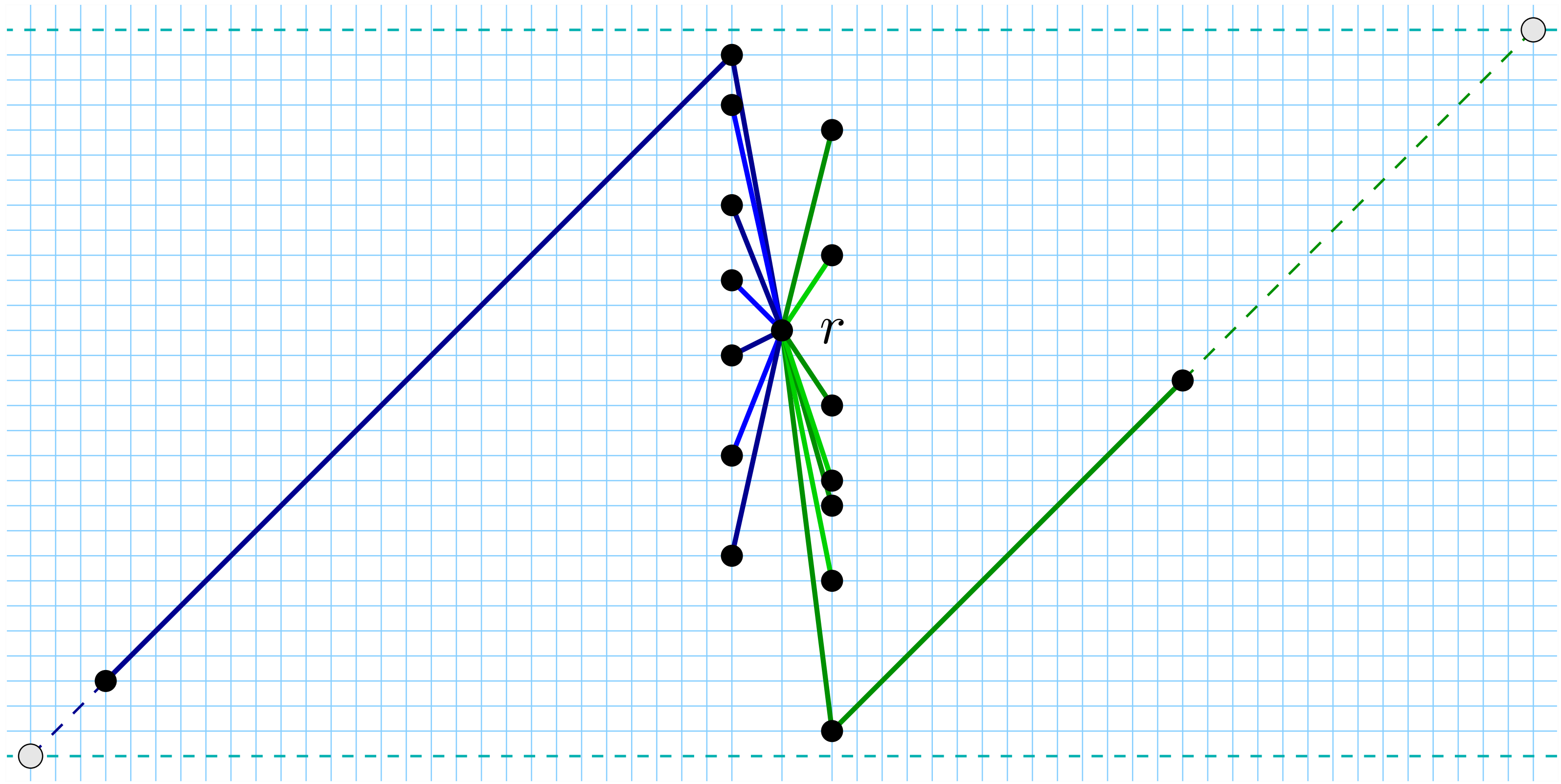

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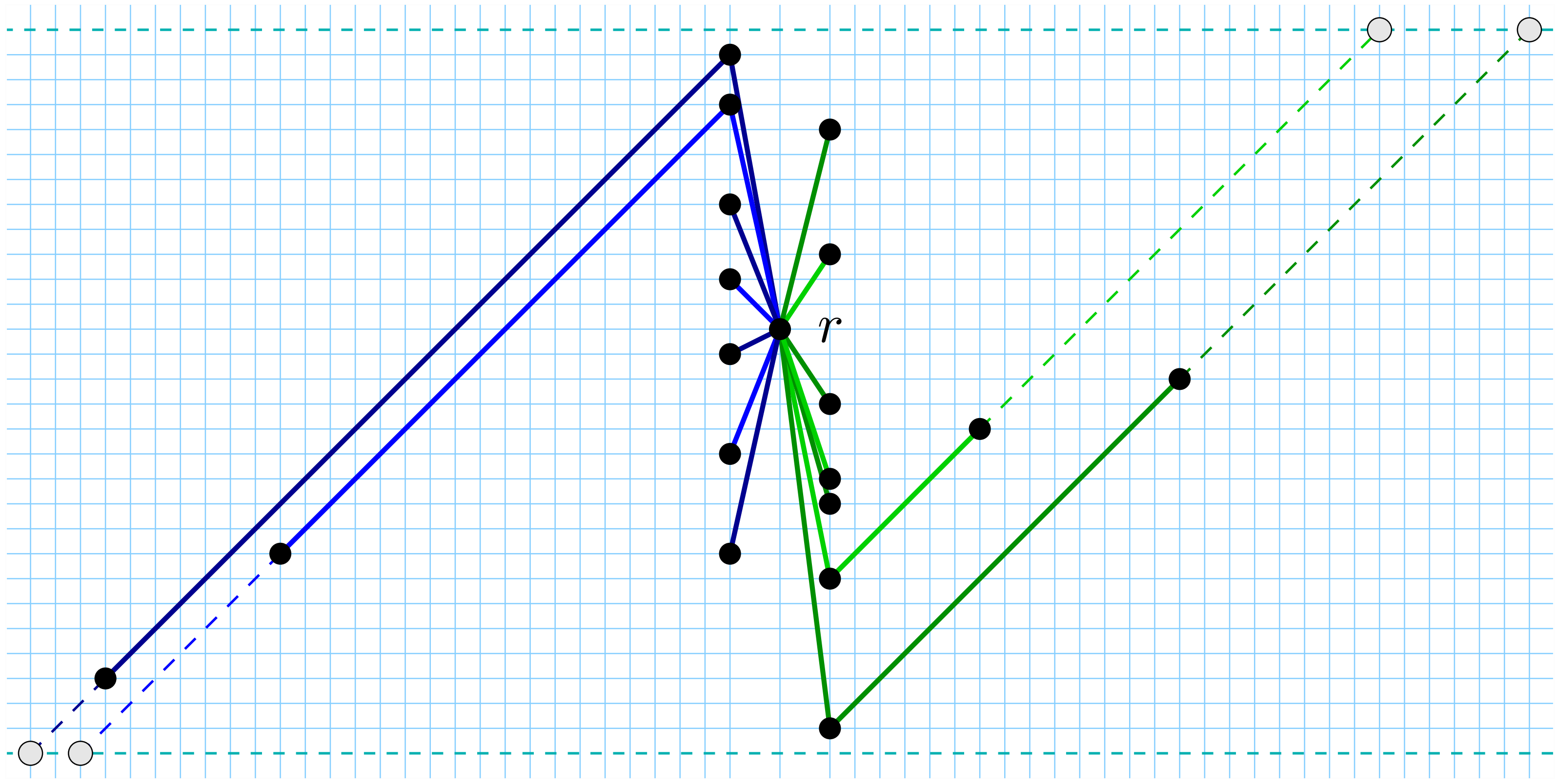

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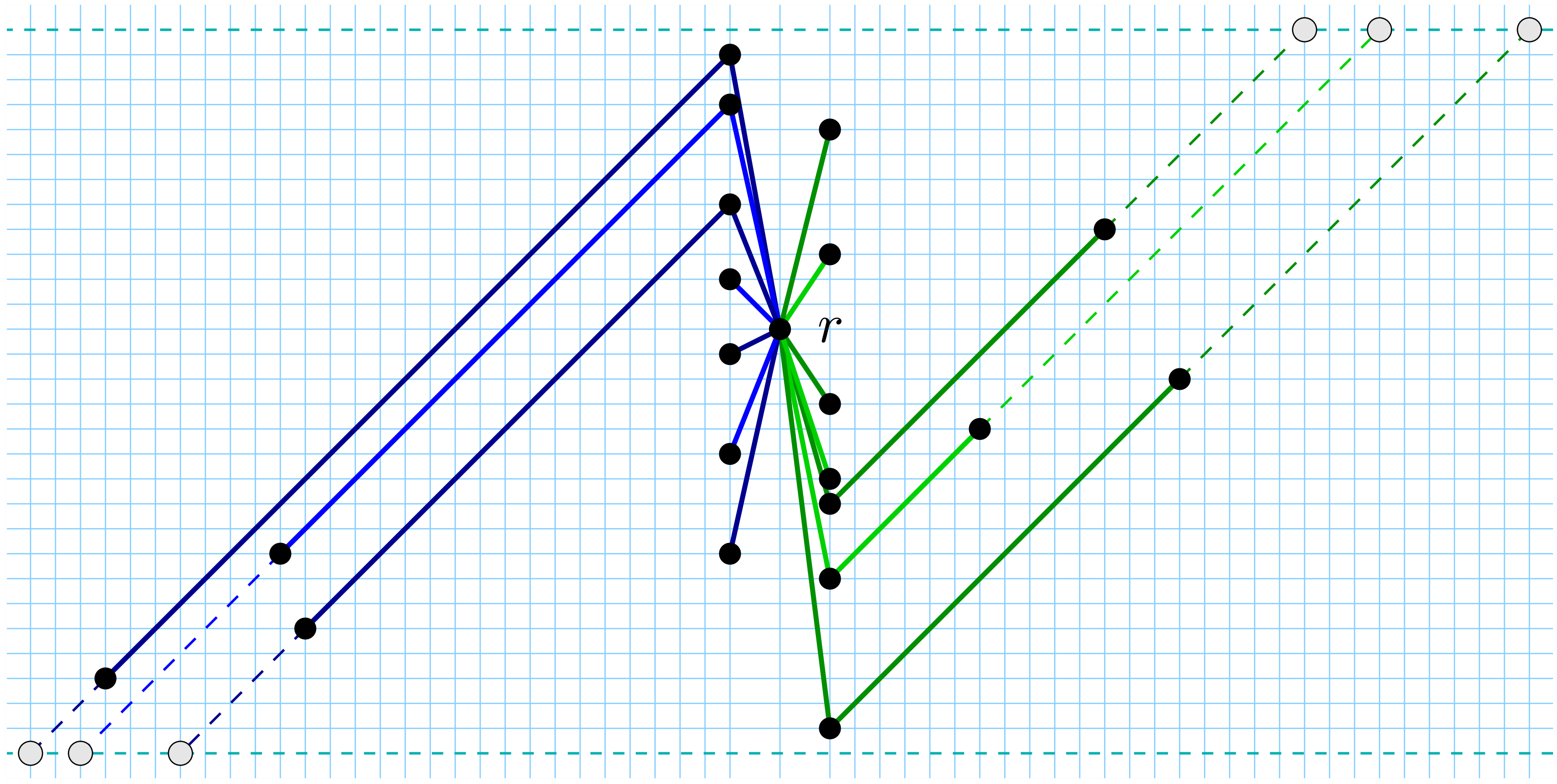

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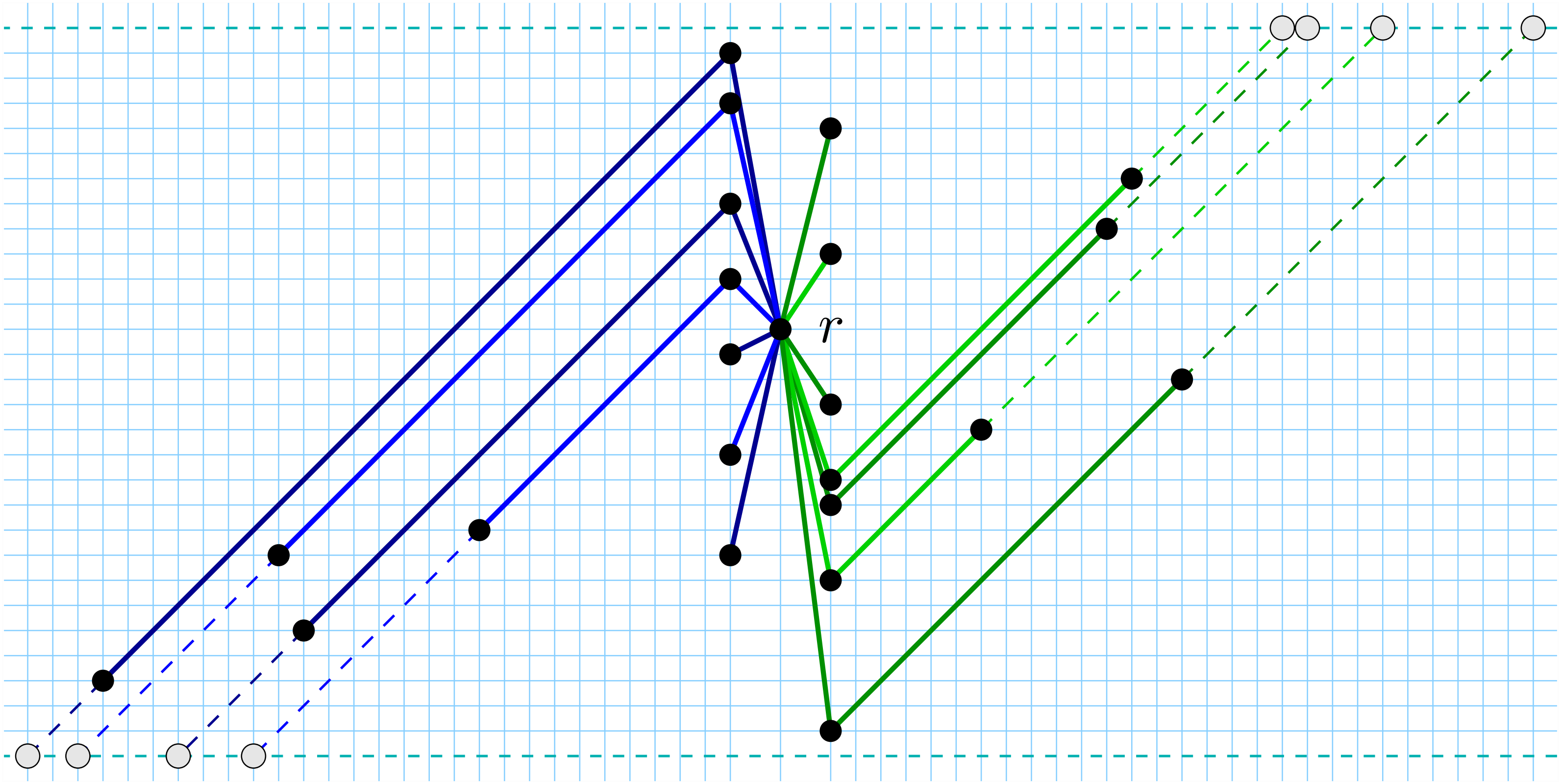

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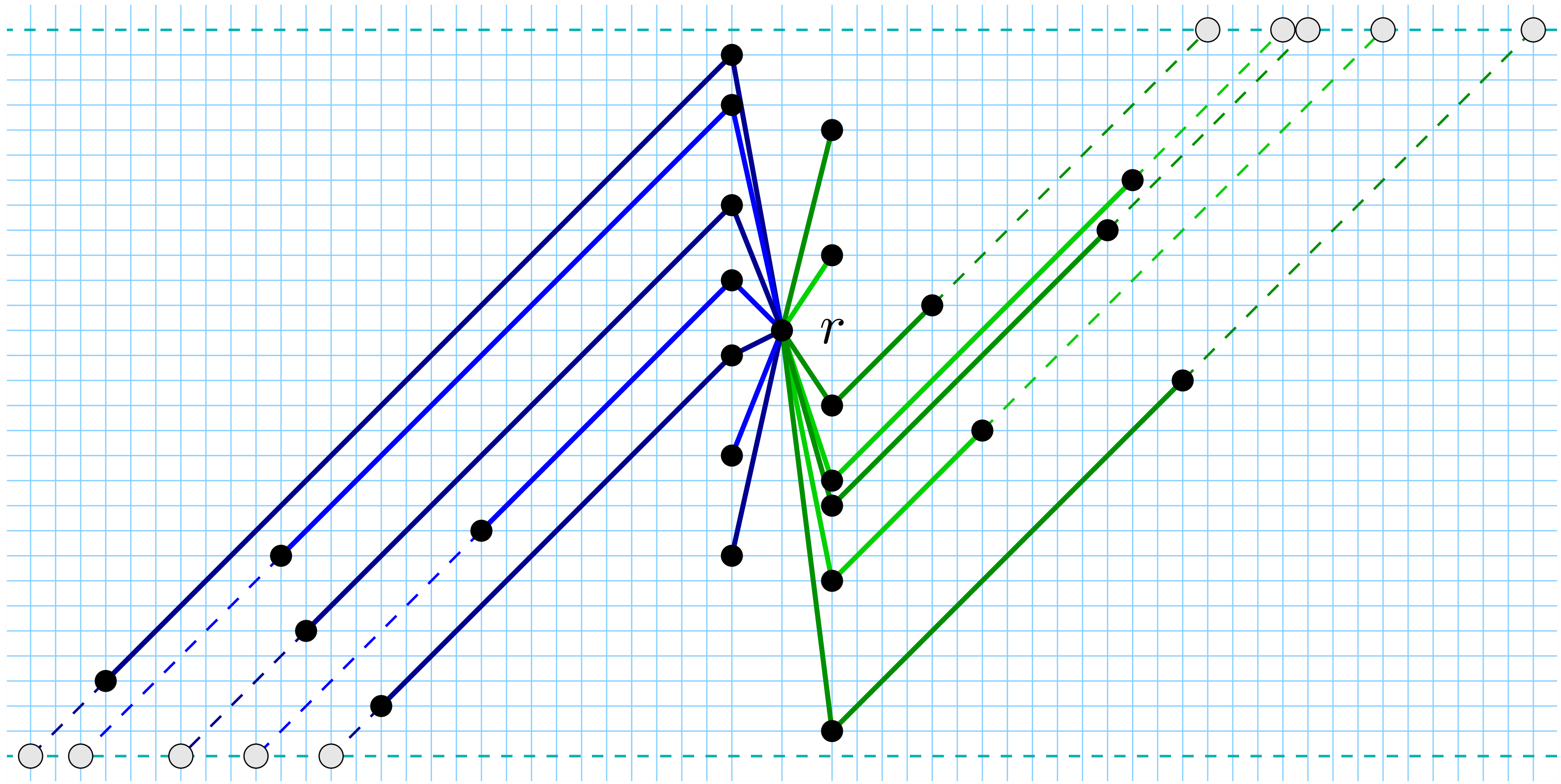

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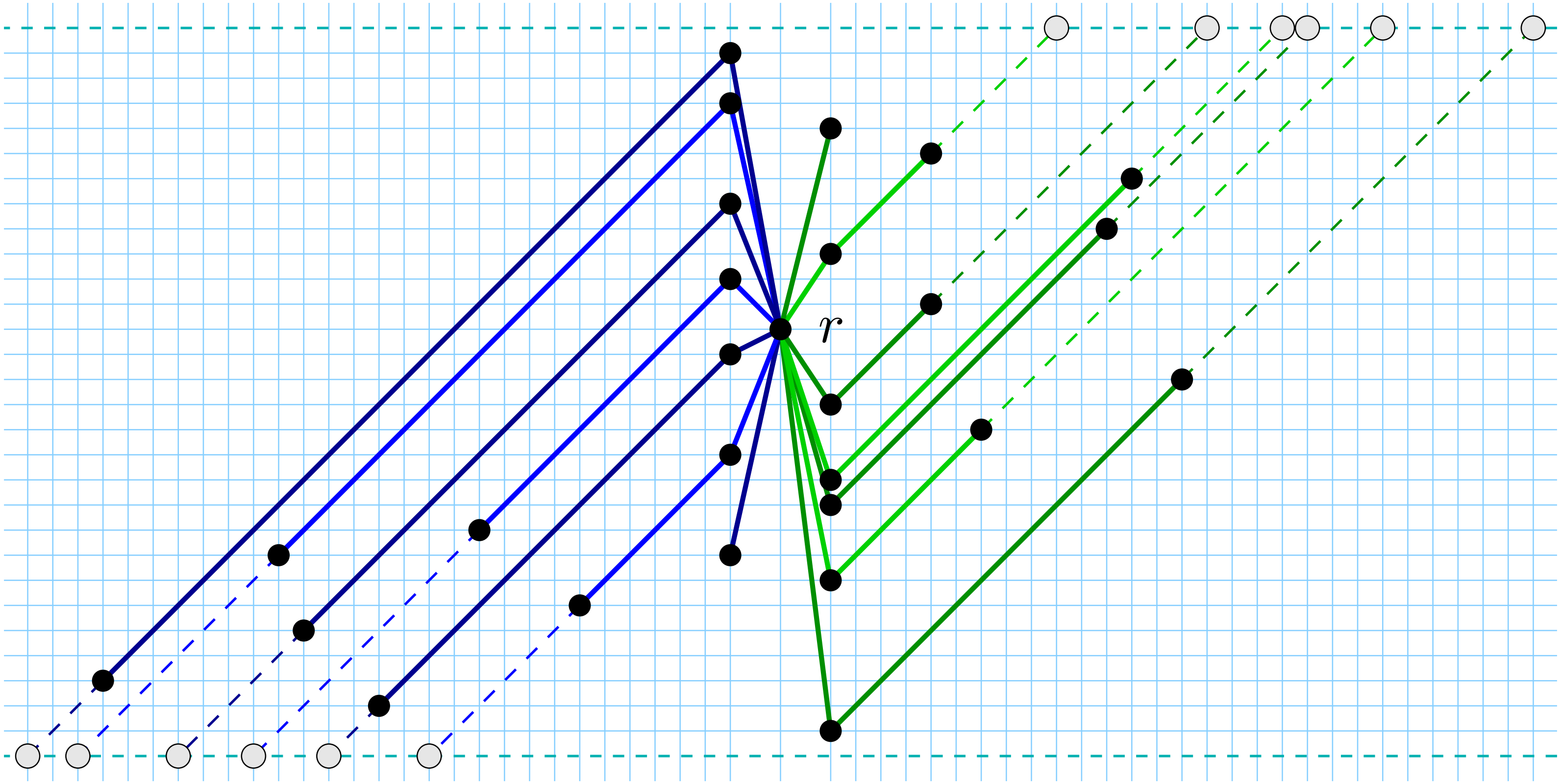

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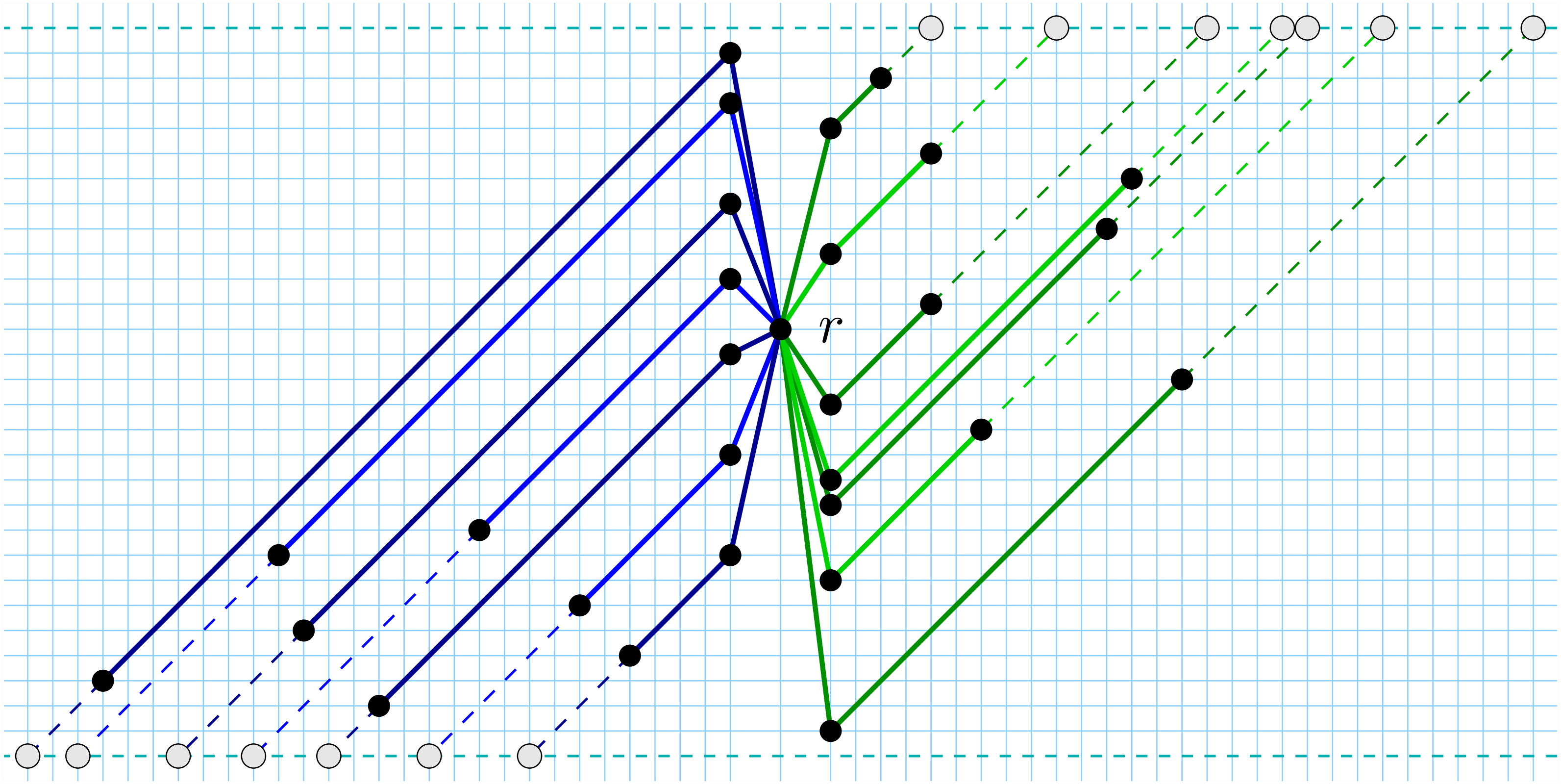

Drawing Algorithms – Degree-3 Spiders

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  - Tricky part is getting degree-3 spider started

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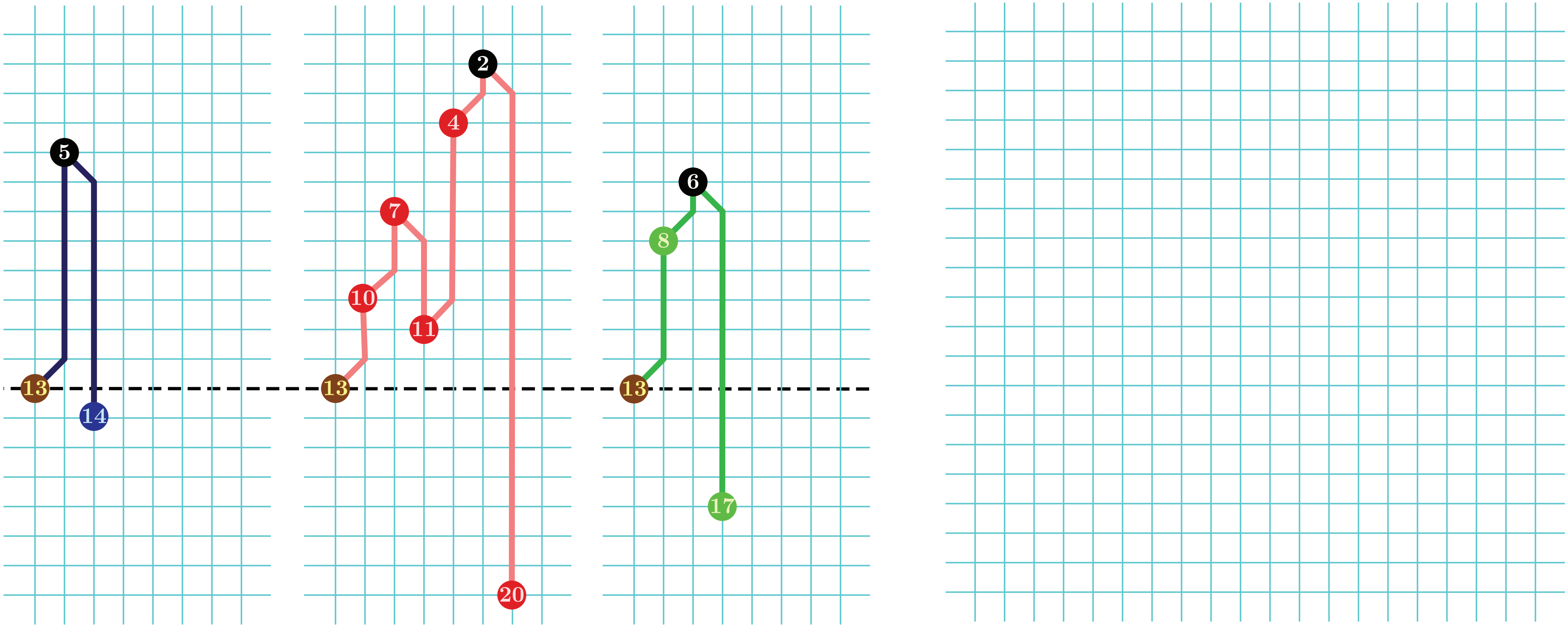

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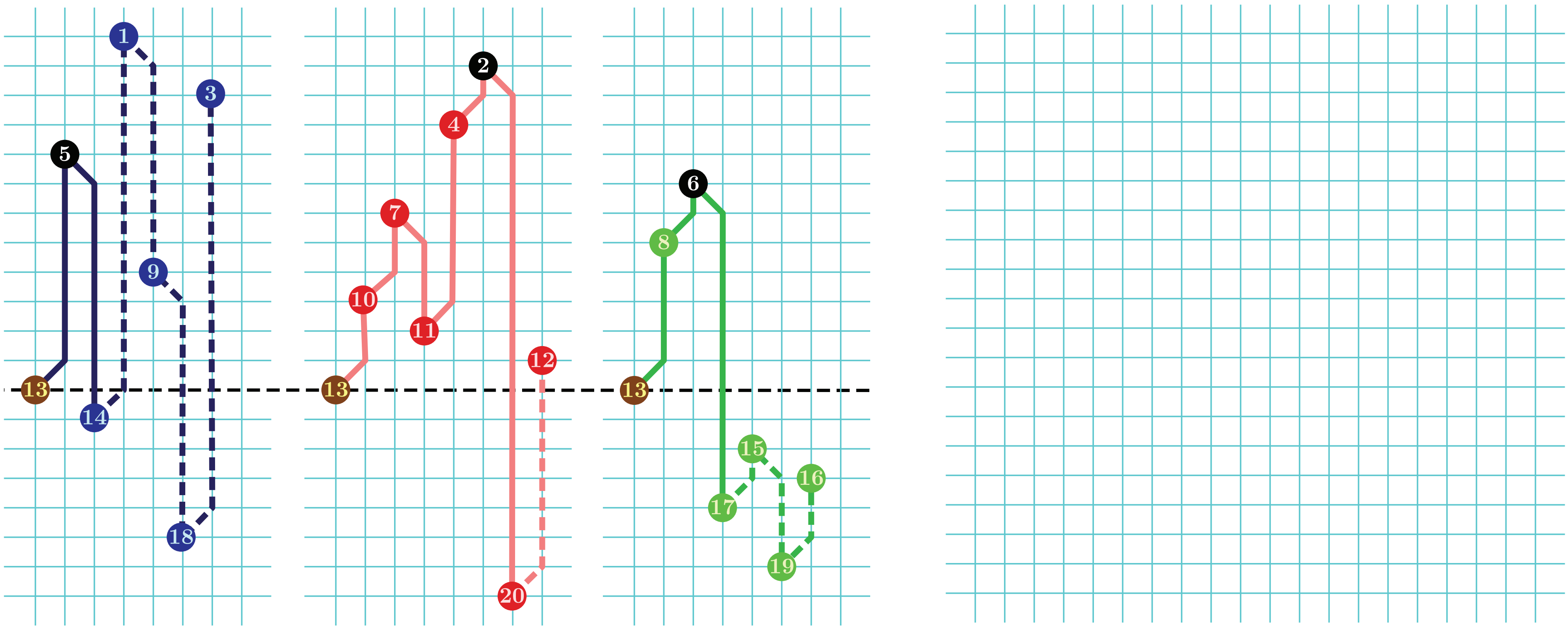

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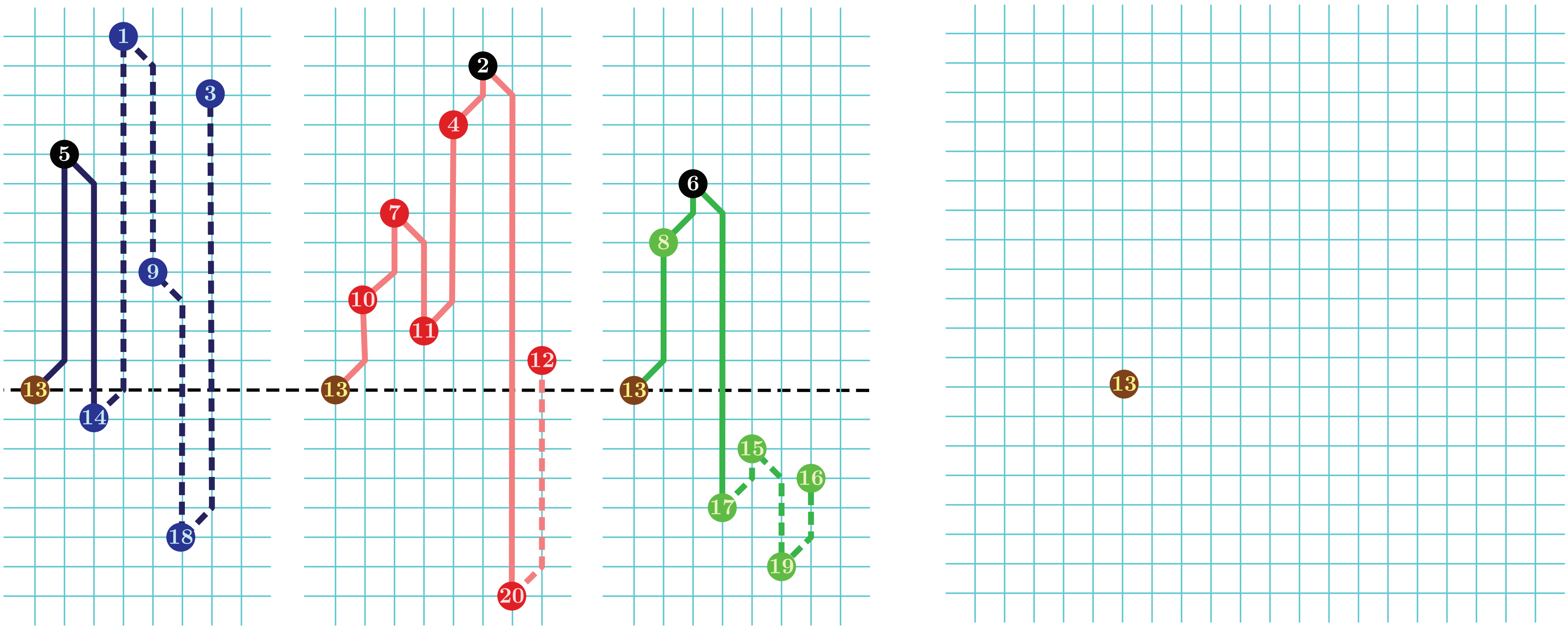

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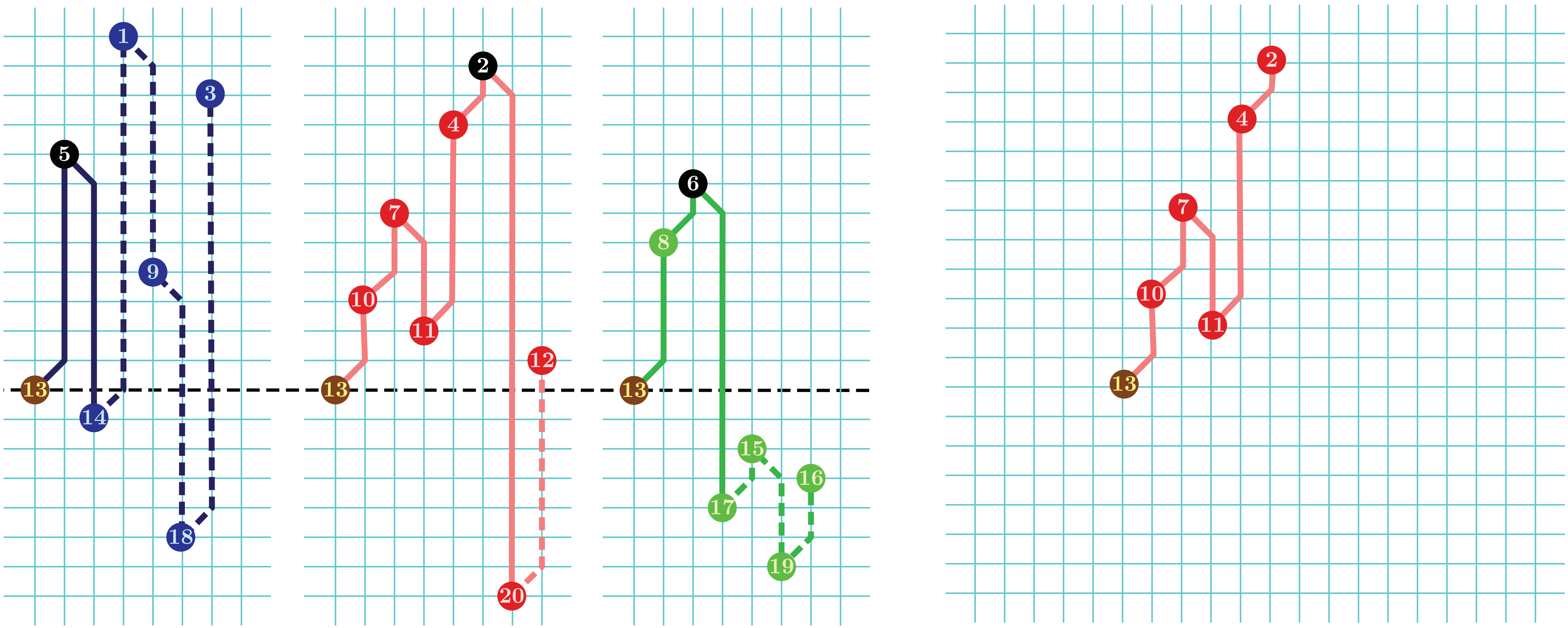

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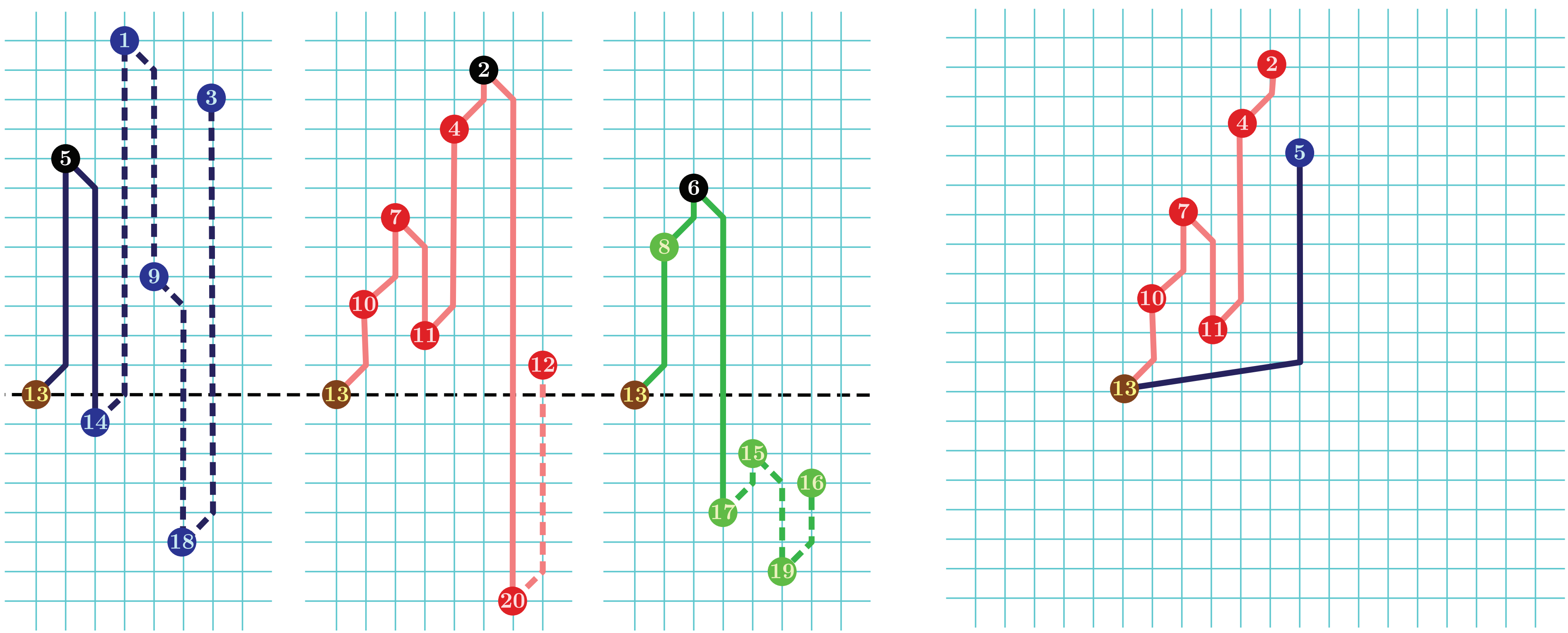

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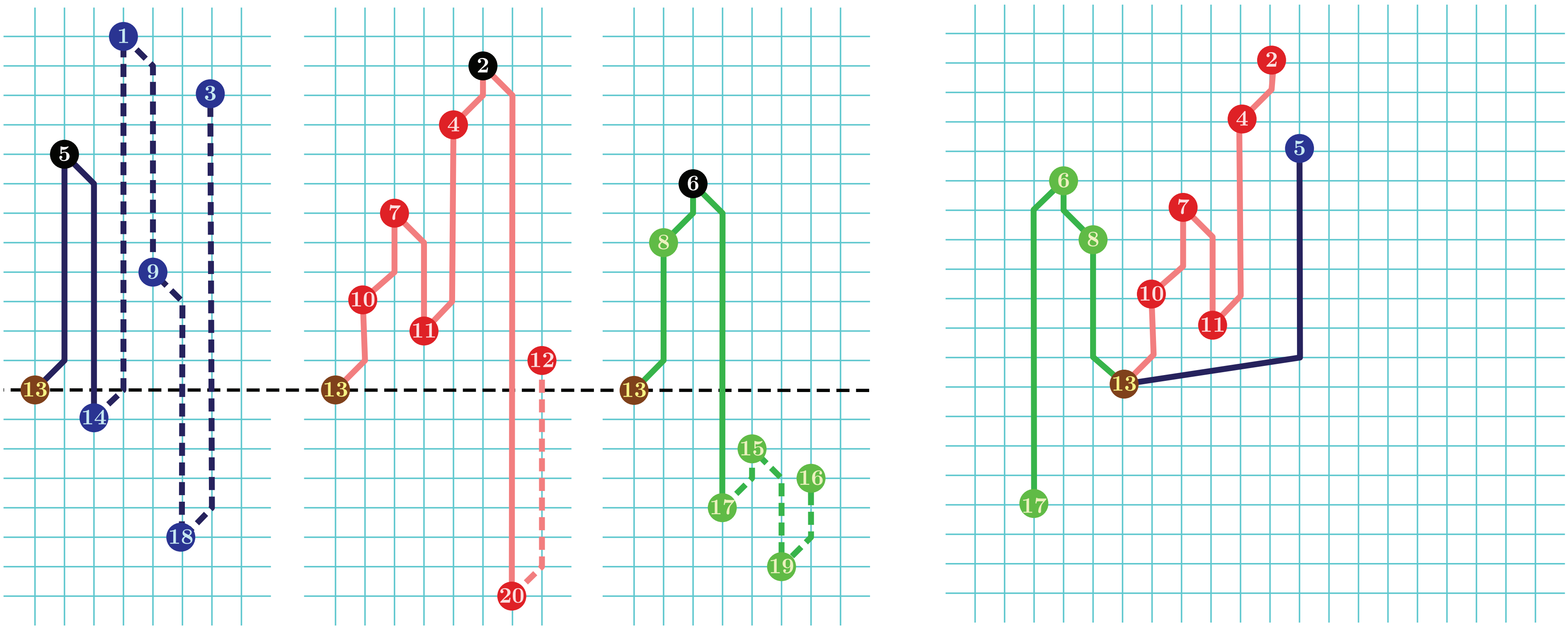

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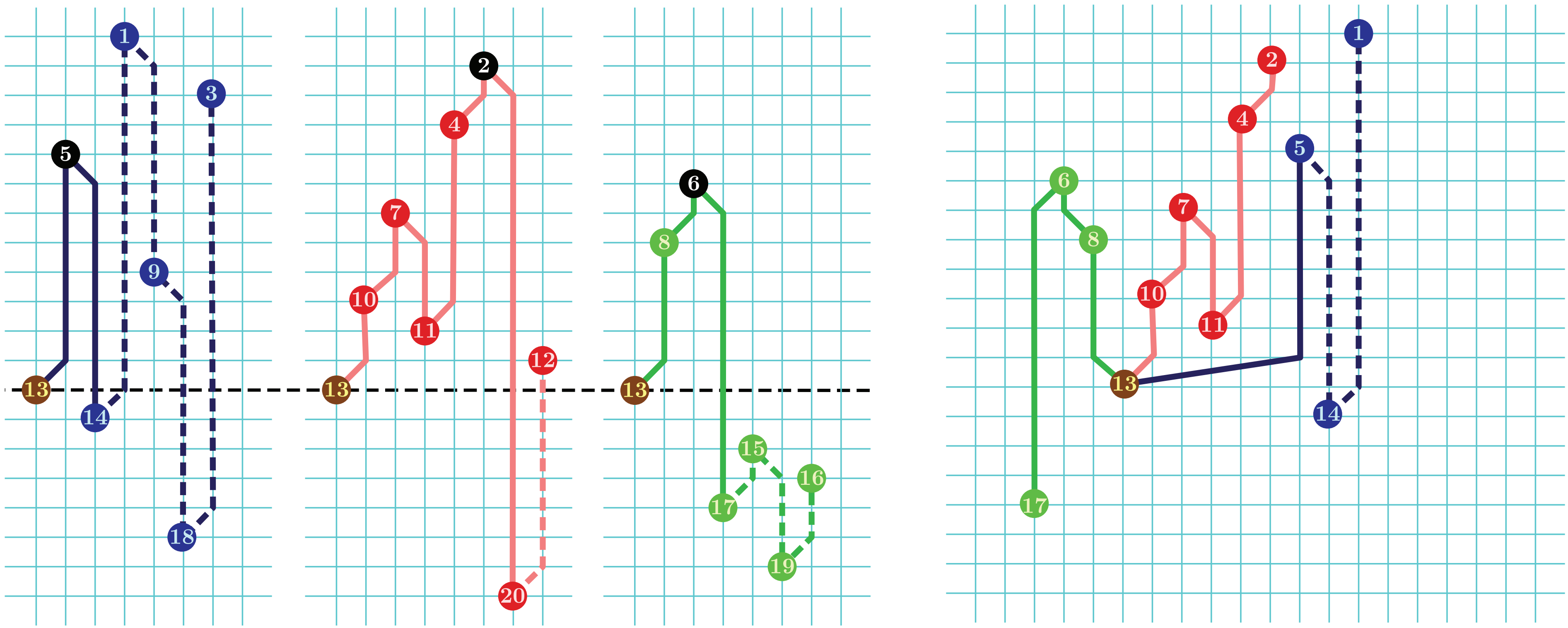

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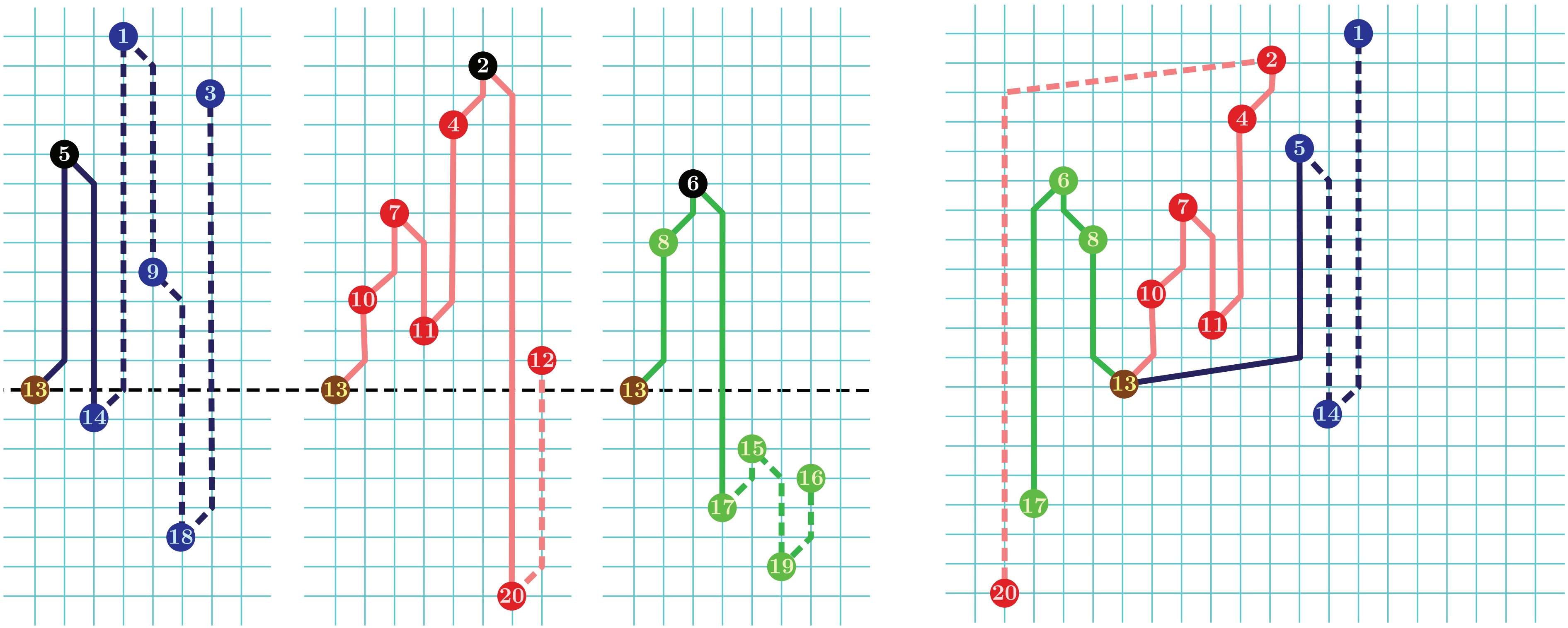

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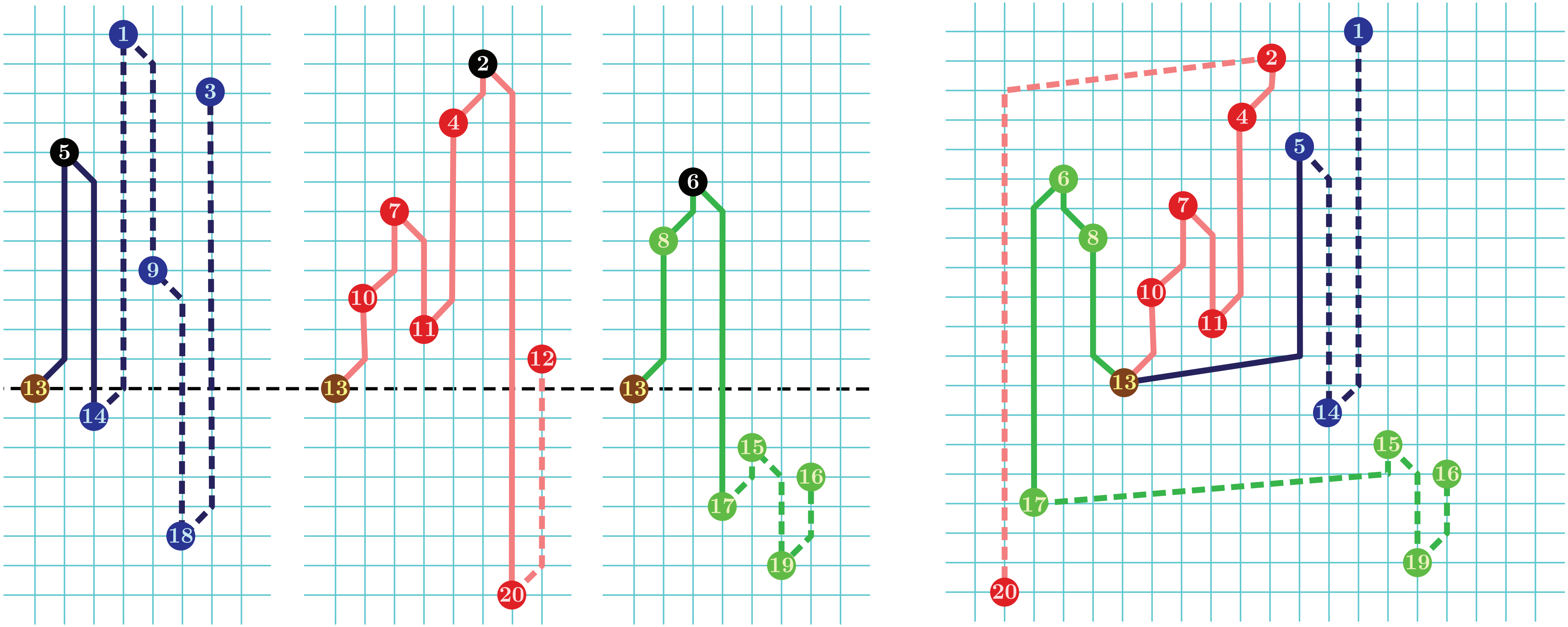

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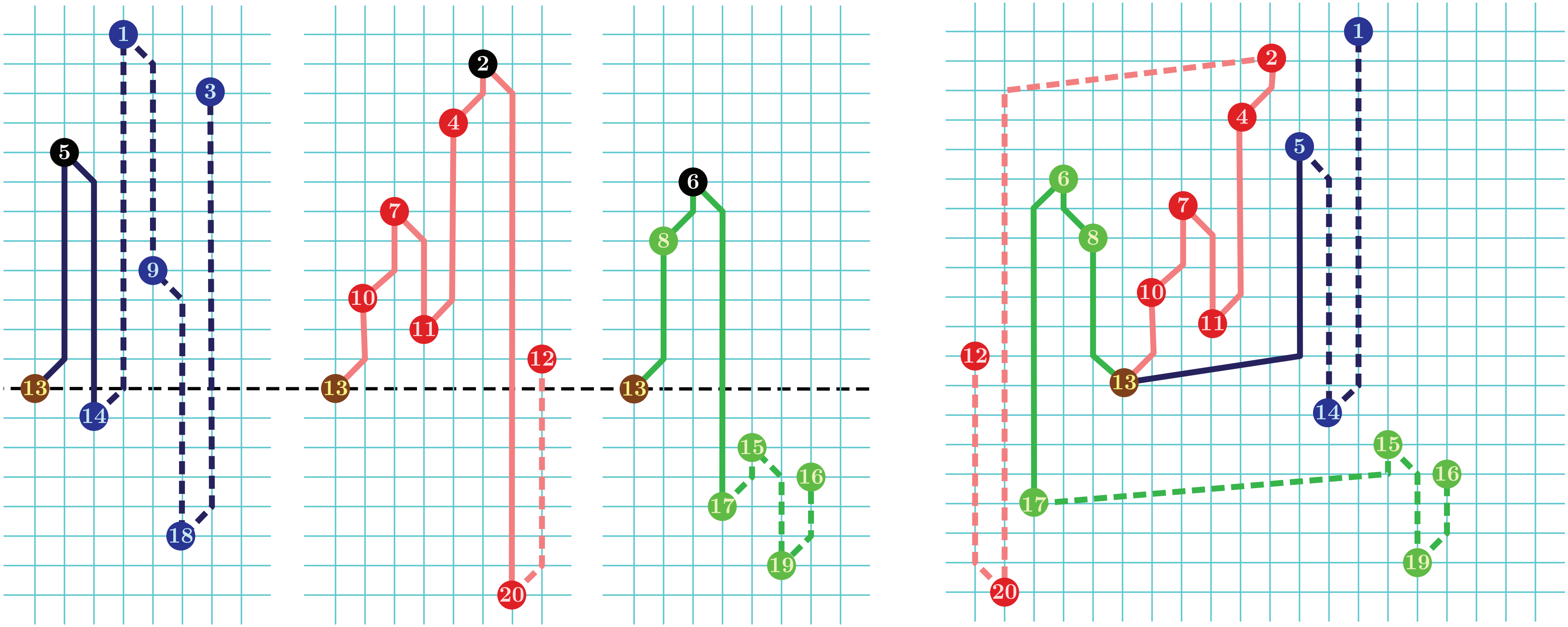

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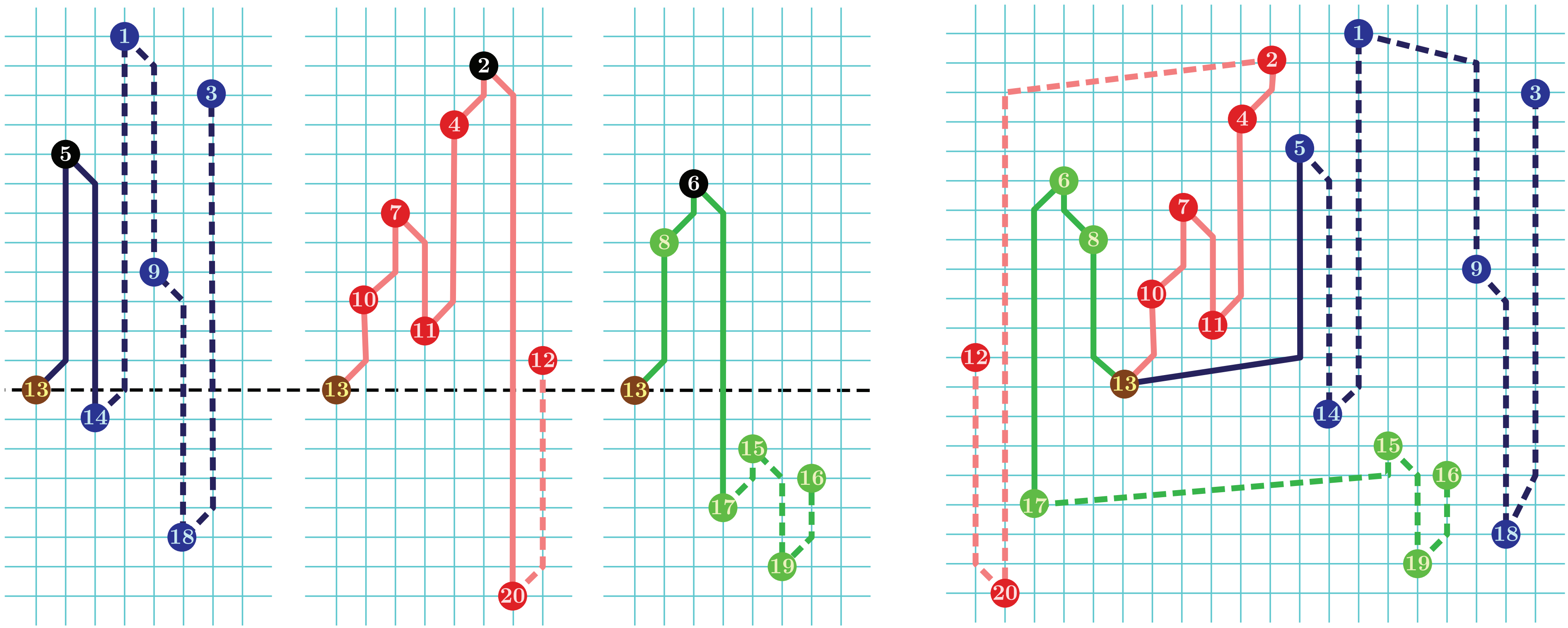

Characterization – ULP Trees

- A minimal lobster argument gives the next theorem:

  Theorem 6 *Every tree either contains a subdivision of* T_8 *or* T_9 *in which case it is not ULP, or it is a caterpillar, a radius-*2 *star, or a degree-*3 *spider in which case it is ULP.*

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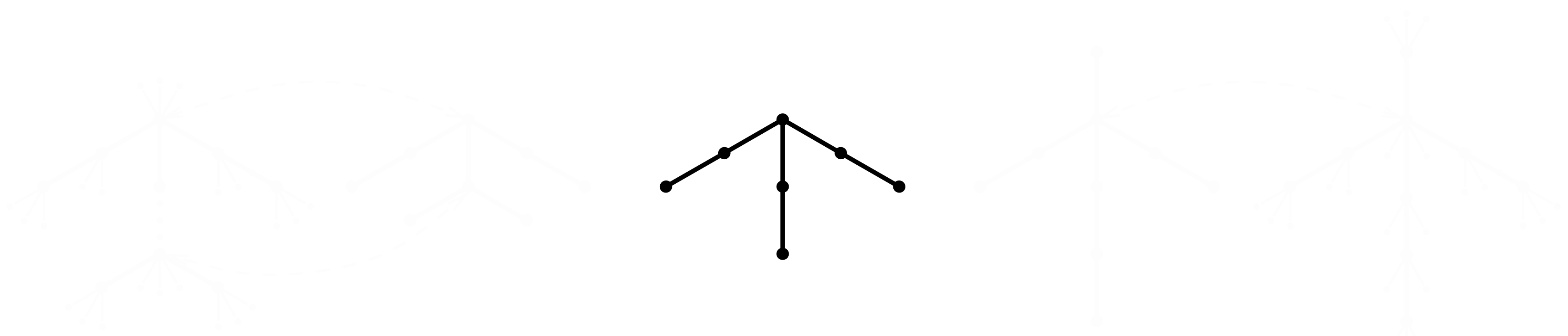
  - A tree that is not a caterpillar has a minimal lobster T_7



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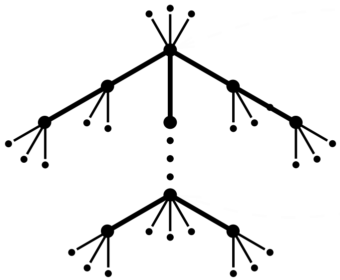
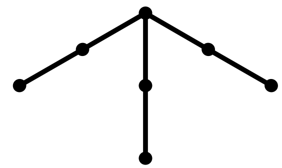
Characterization – ULP Trees





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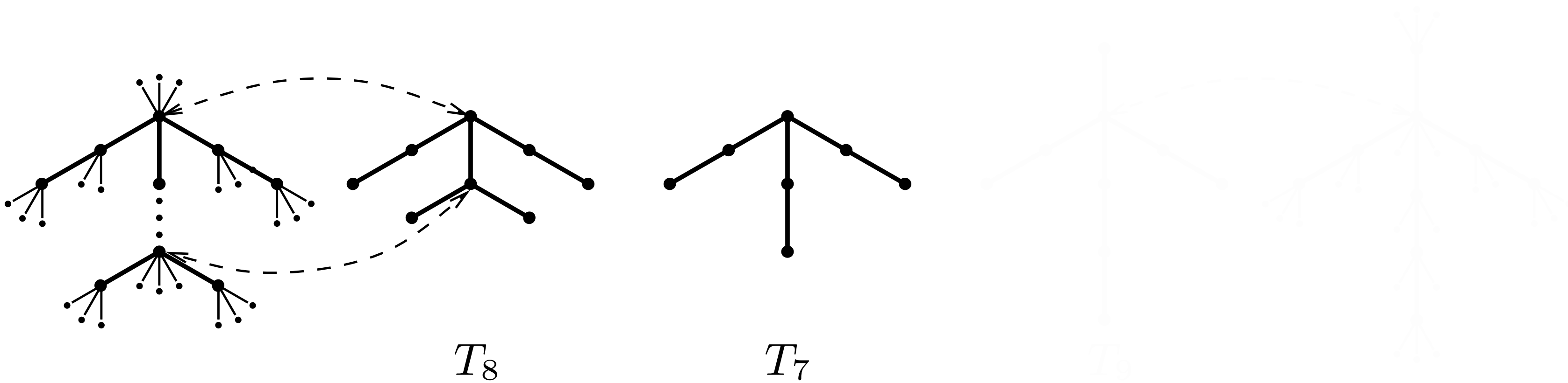

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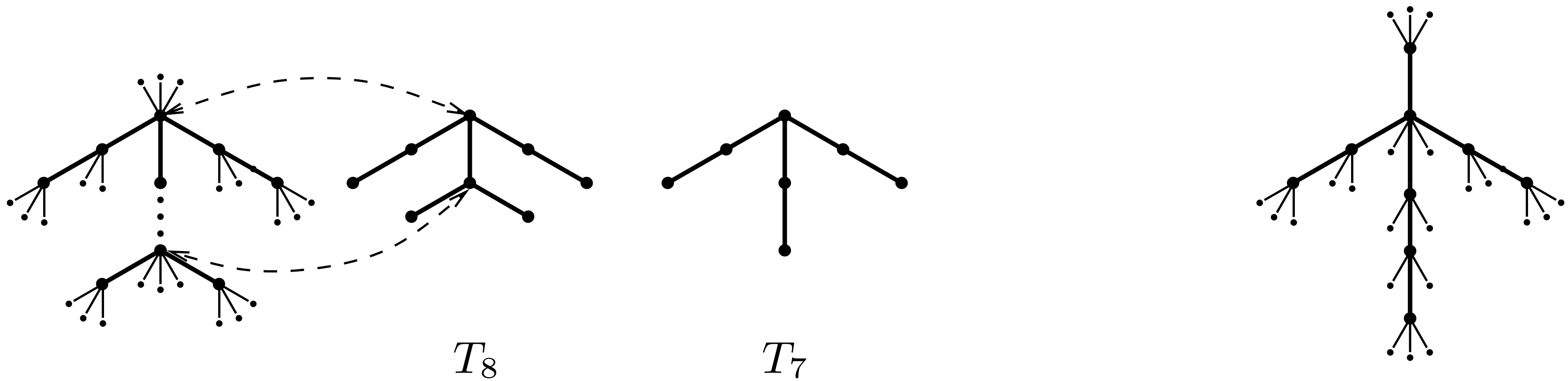
Characterization – ULP Trees



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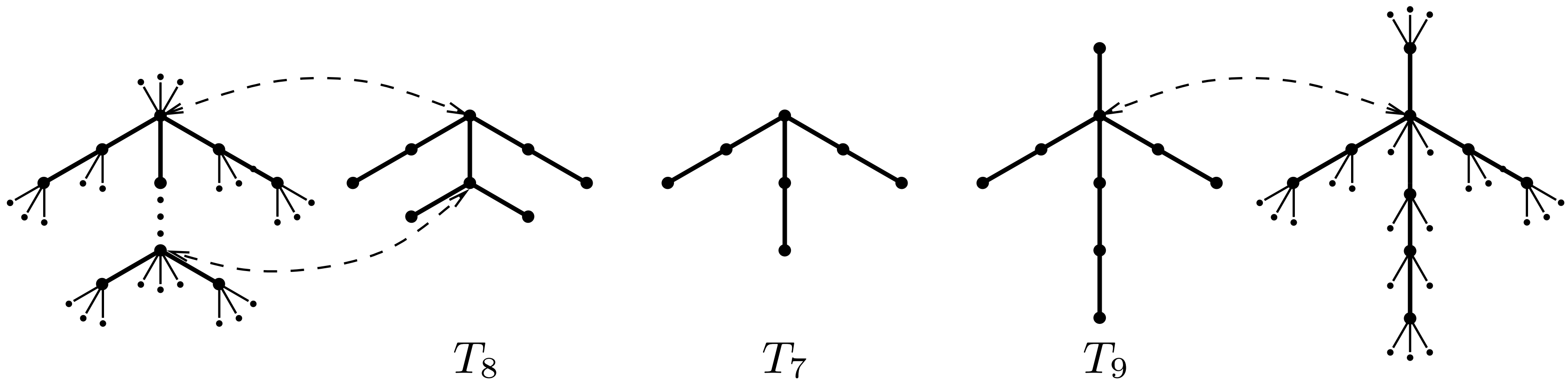
Characterization – ULP Trees



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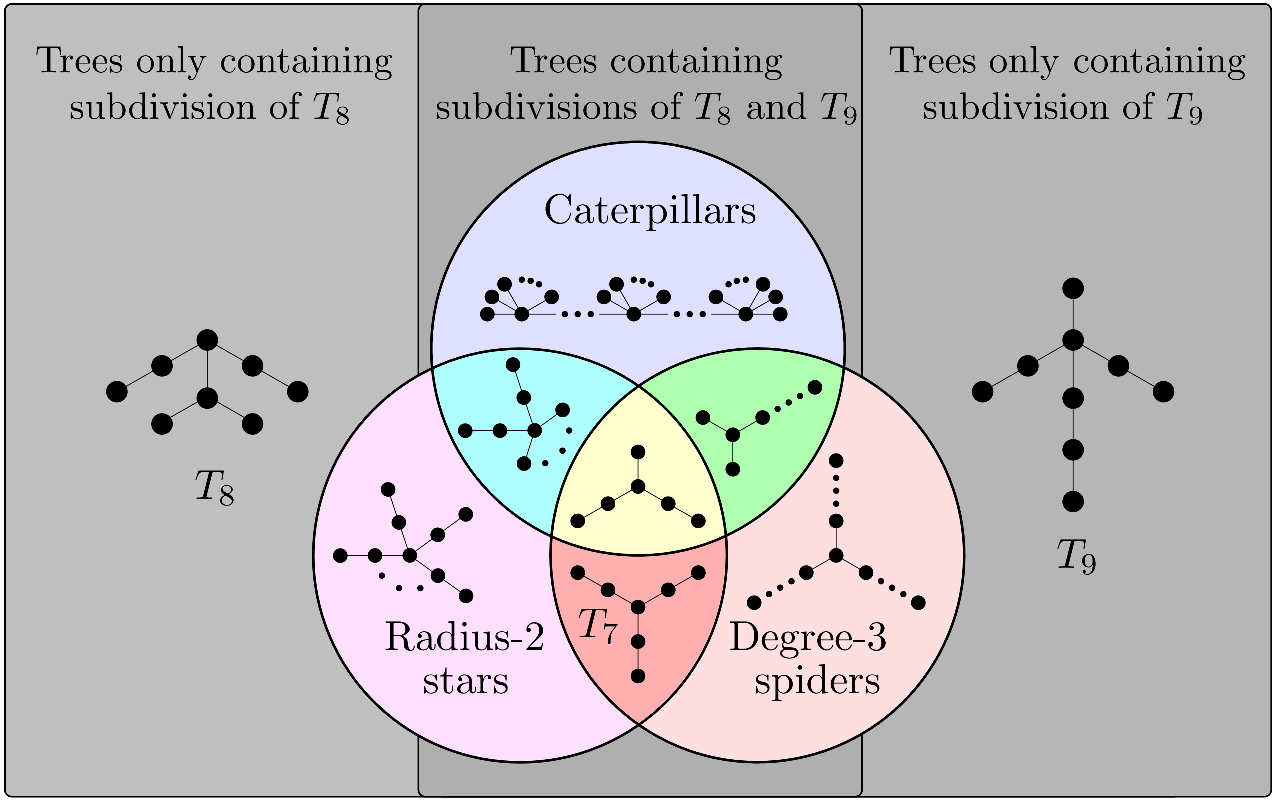
    - Has at least one vertex of degree-4 – contains T_9

Outline – Unlabeled Level Planar Trees

- Background
- Unlabeled Level Planar Trees



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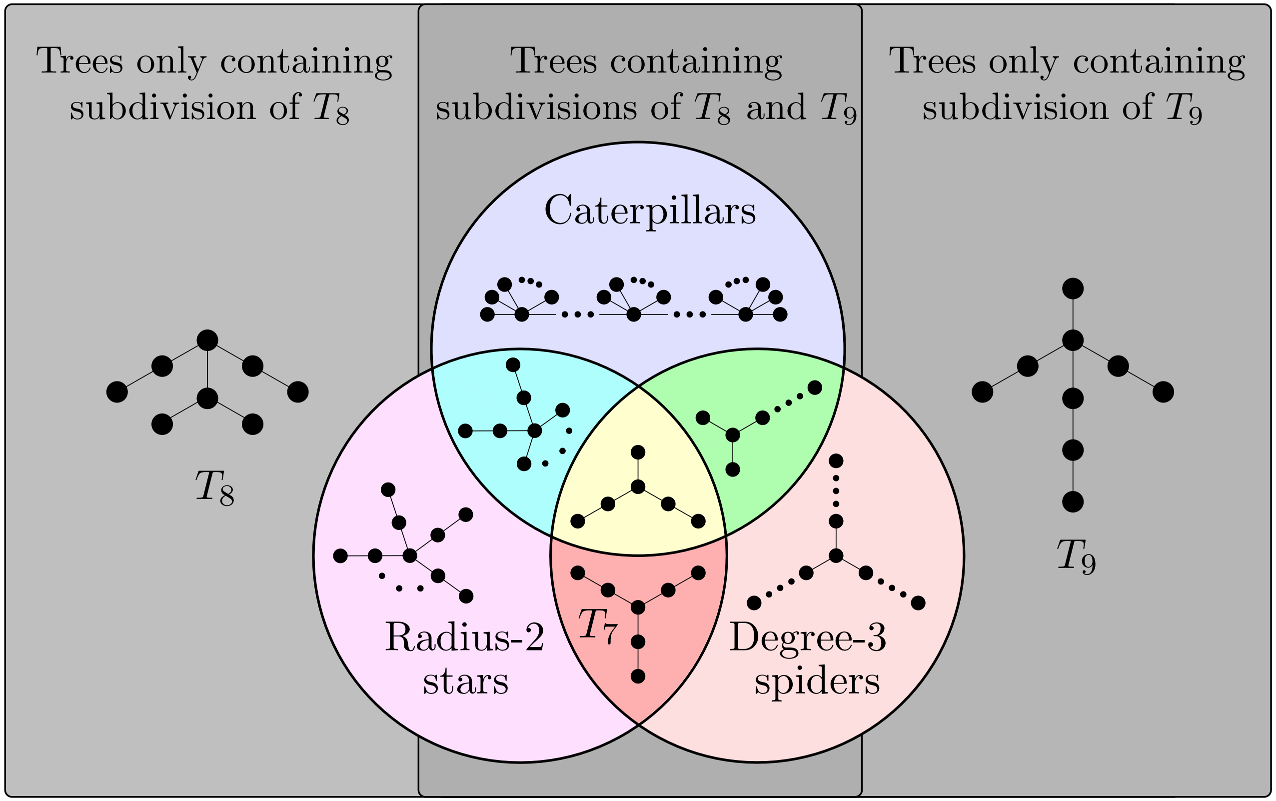

- First showed T_8 and T_9 are not ULP

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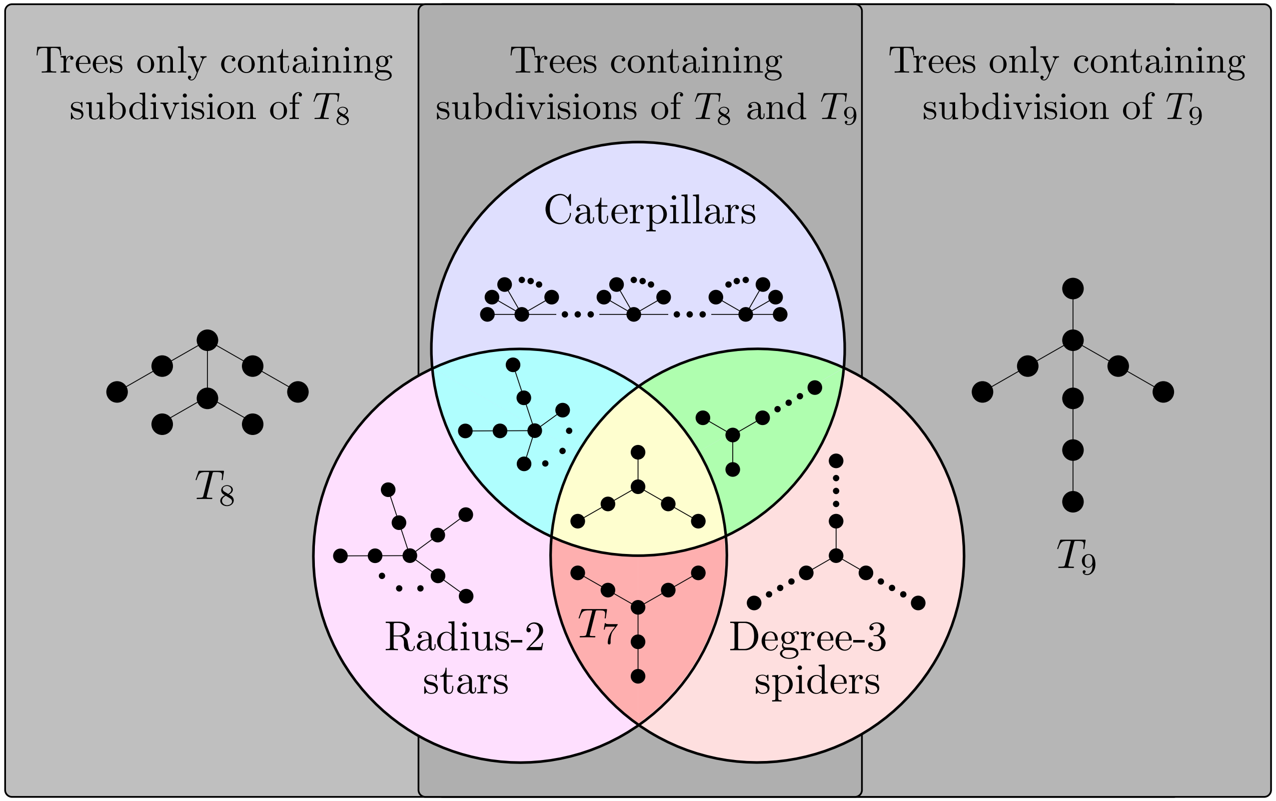

  - Showed caterpillars, radius-2 stars and degree-3 spiders are ULP

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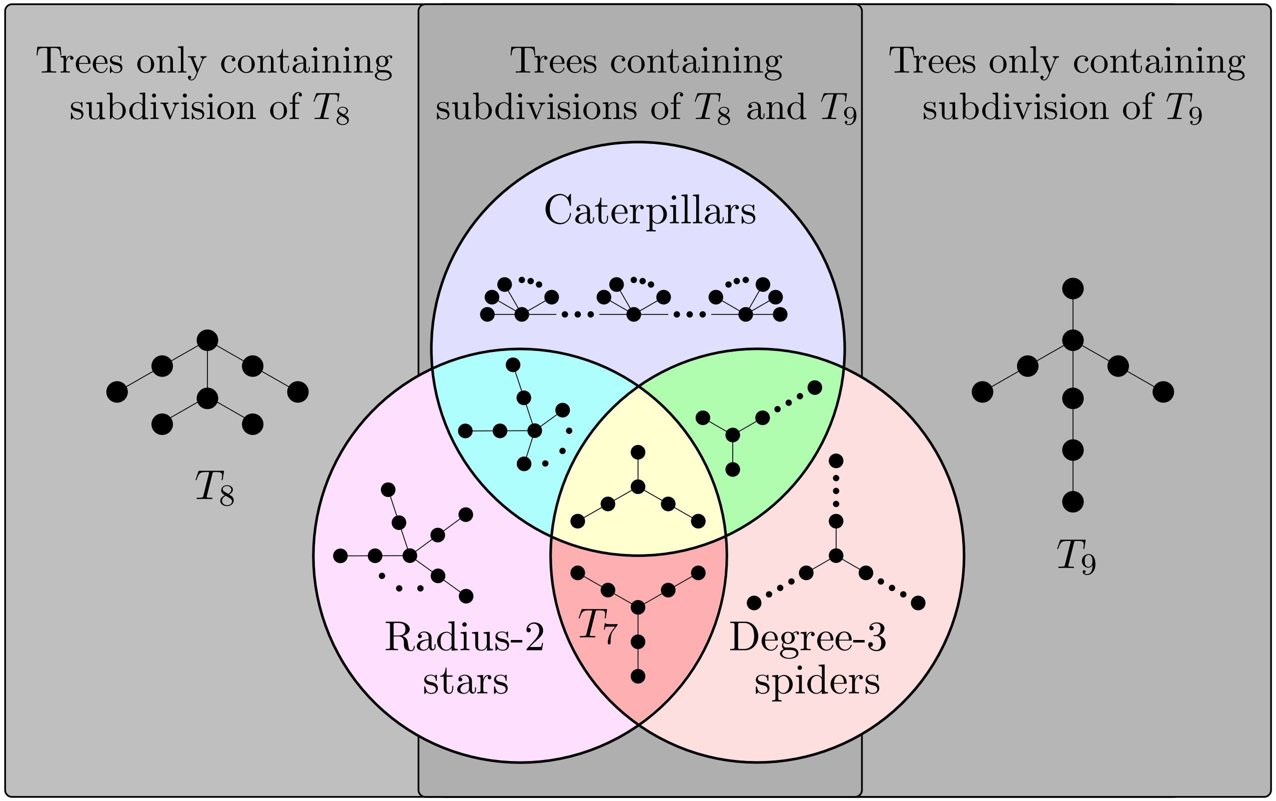

- Last showed all trees fall into one the above categories

Outline – Unlabeled Level Planar Graphs

- Background
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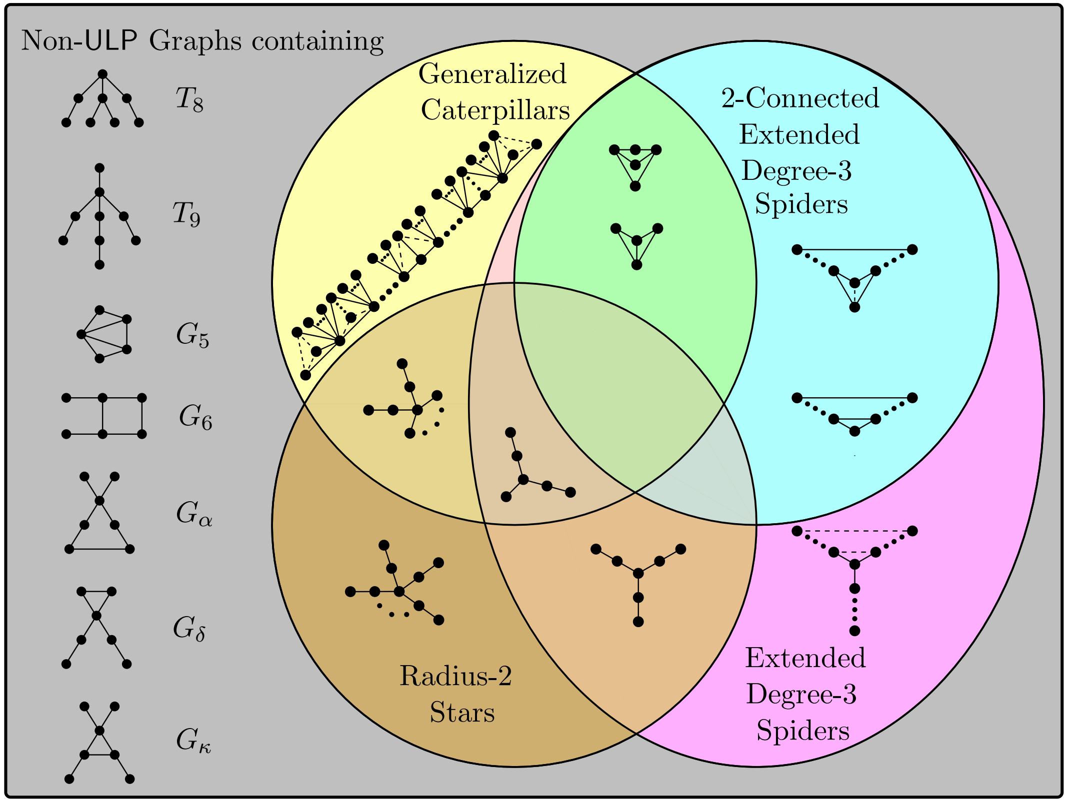
- Unlabeled Level Planar Graphs



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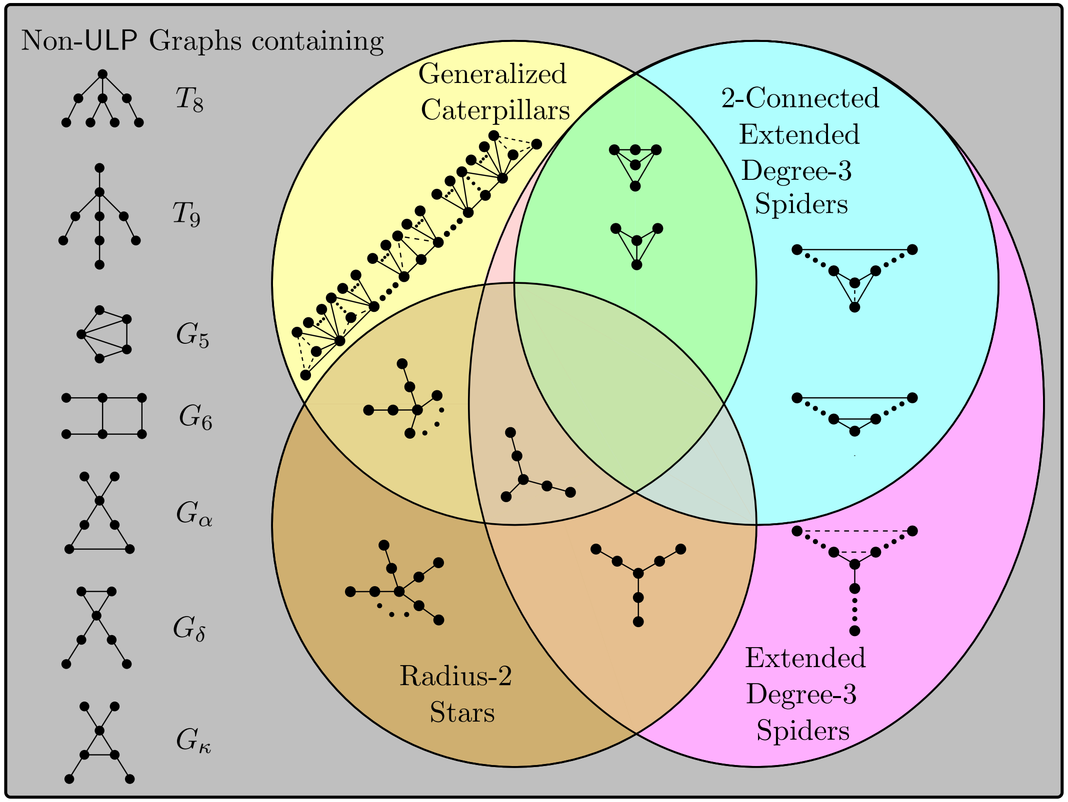
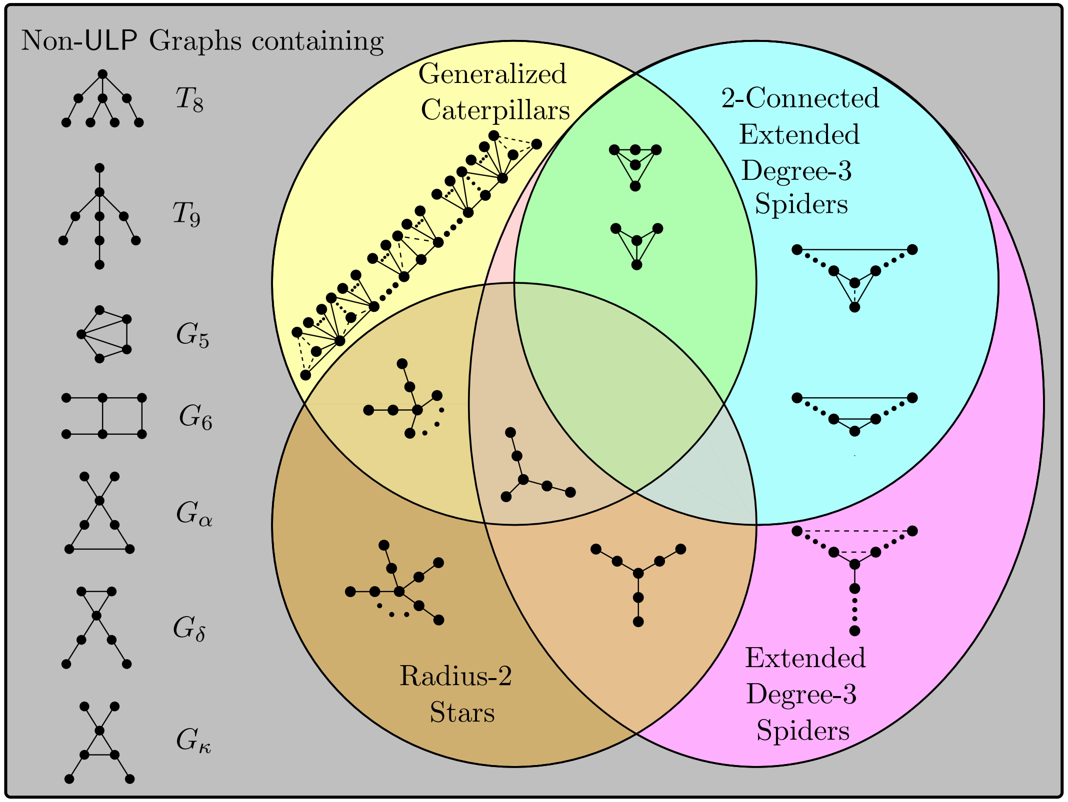


- First show forbidden graphs are not ULP

- Background
- Unlabeled Level Planar Trees
- Unlabeled Level Planar Graphs



  - ► Then extend drawing for ULP graphs with cycles

Outline – Unlabeled Level Planar Graphs

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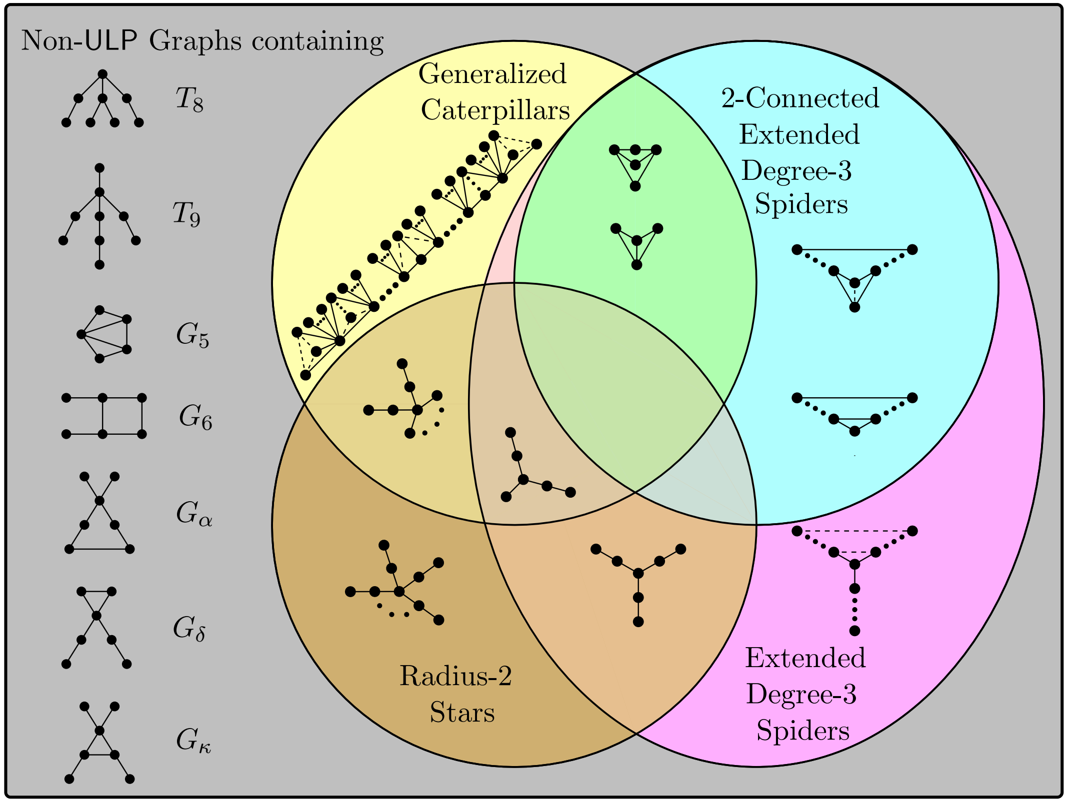

- Finally describe how all graphs fall into one of the above categories

Forbidden Graphs – G_5, G_6, G_α, G_κ, G_δ

- For each ULP graph let C be the chain a—b—c—d—e

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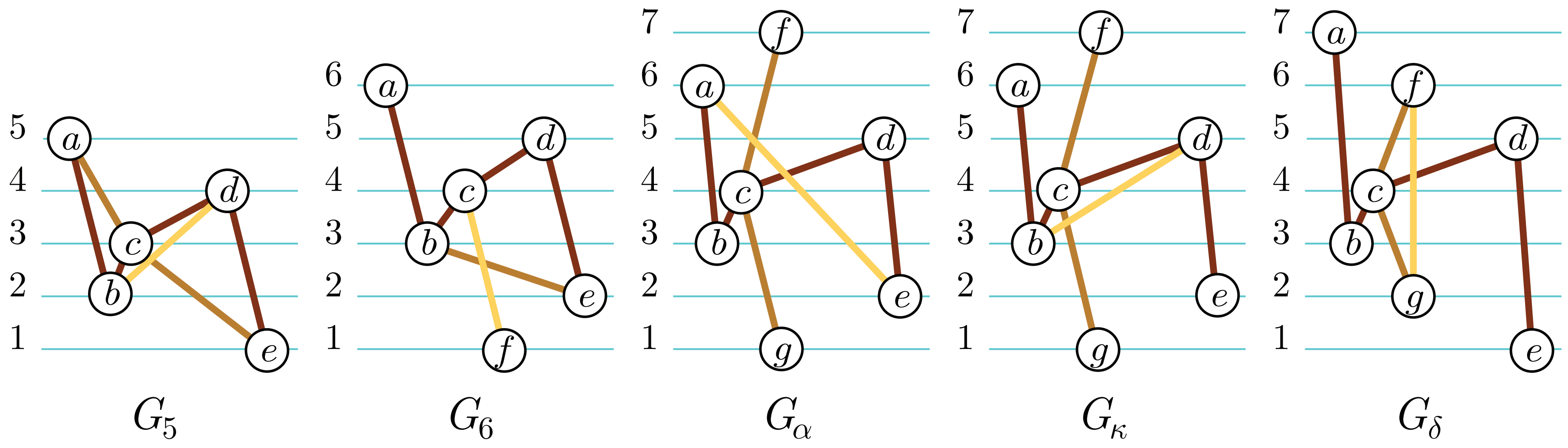

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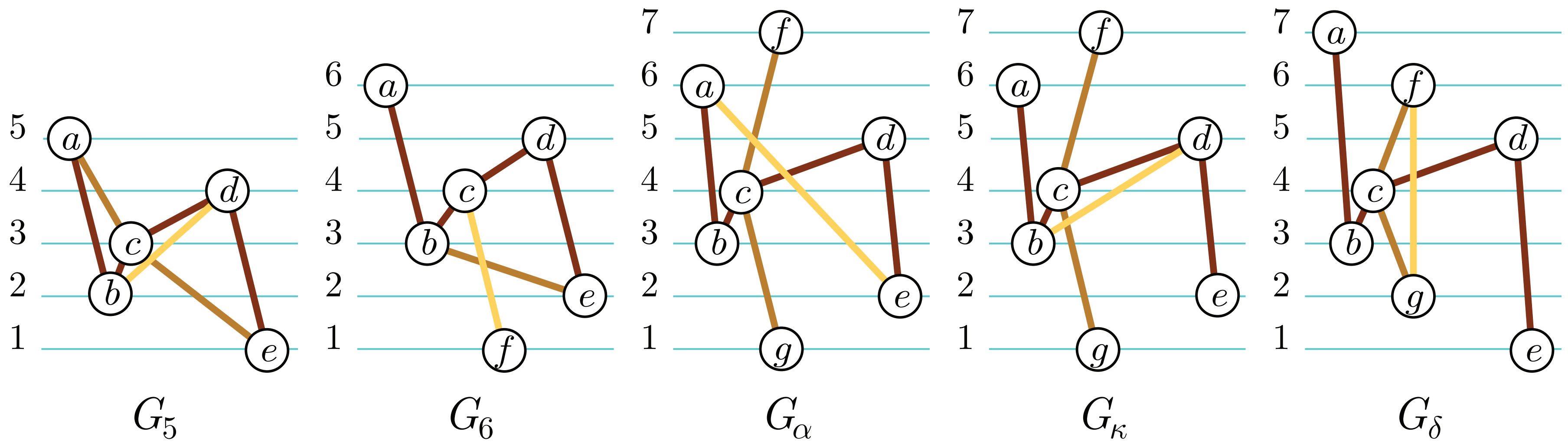

  - Use similar arguments as with T_8 and T_9

Forbidden Graphs – G_5, G_6, G_α, G_κ, G_δ

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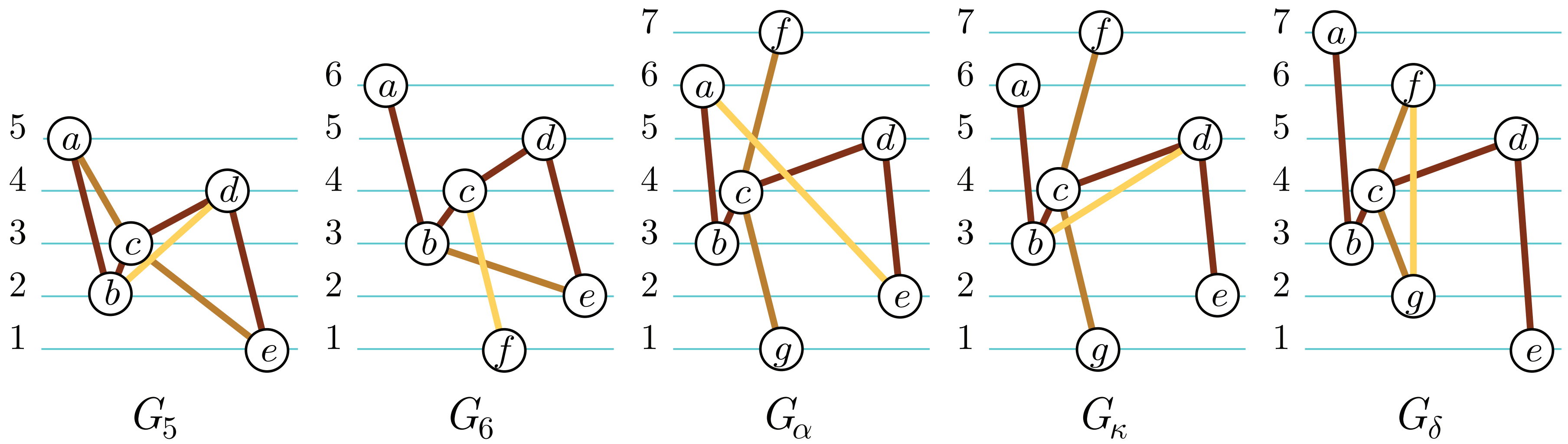

  - Implies one of the other edges must cross C

Forbidden Graphs – $G_5, G_6, G_\alpha, G_\kappa, G_\delta$

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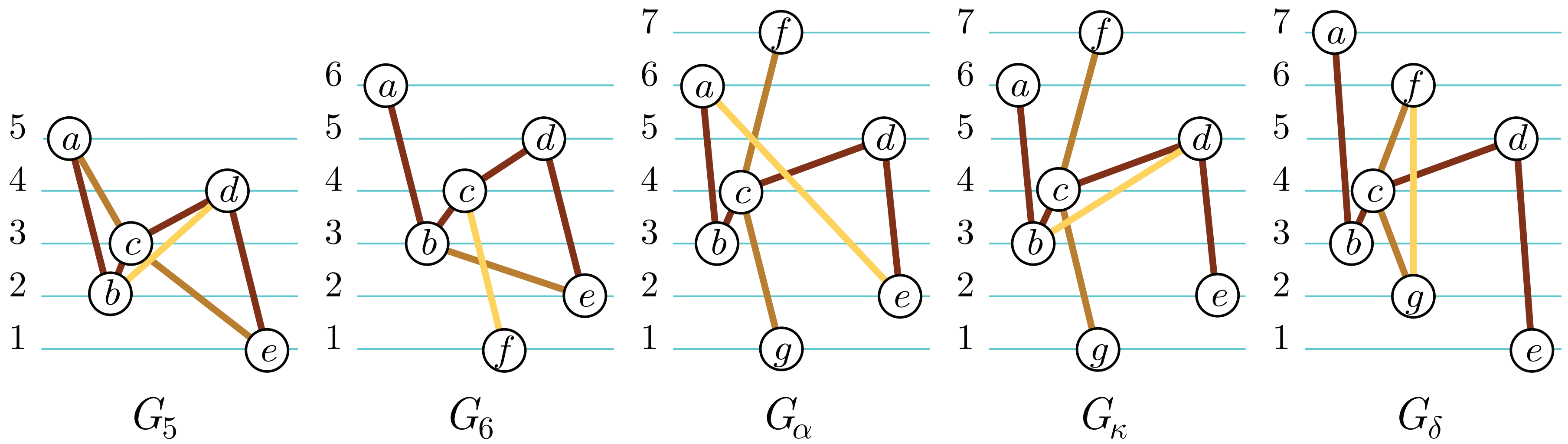
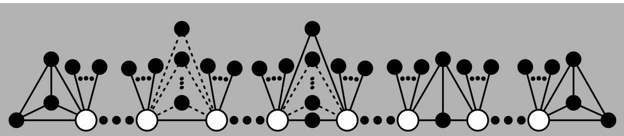


Lemma 7 *Any graph with a cycle containing a subdivision of G_5, G_6, G_α, G_κ, or G_δ cannot be ULP.*

Generalized Caterpillar



- Drawing a generalized caterpillar in linear time:

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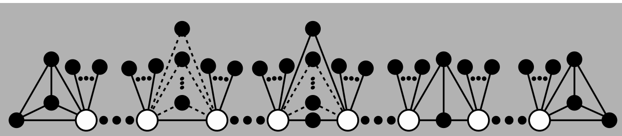

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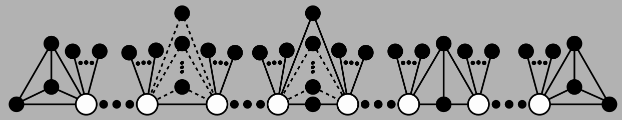

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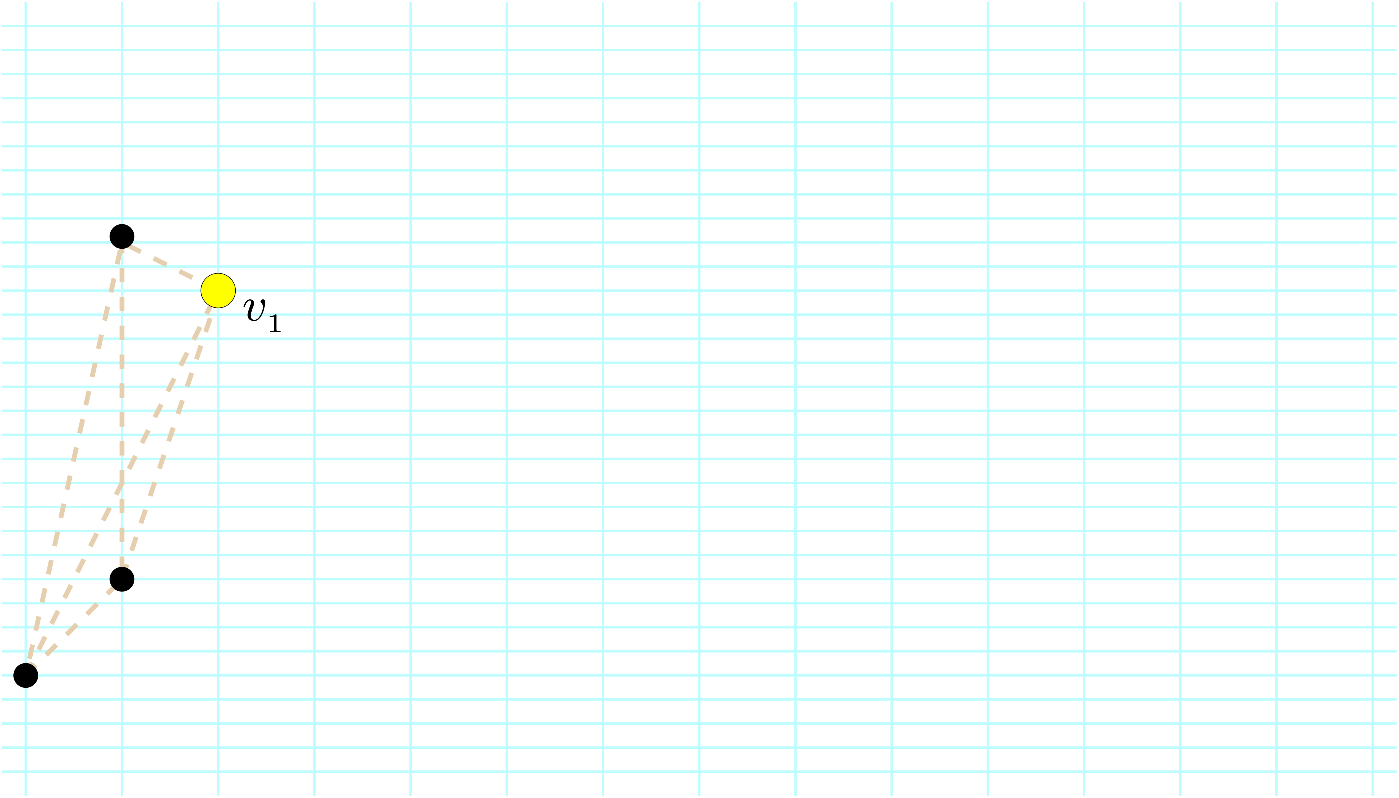

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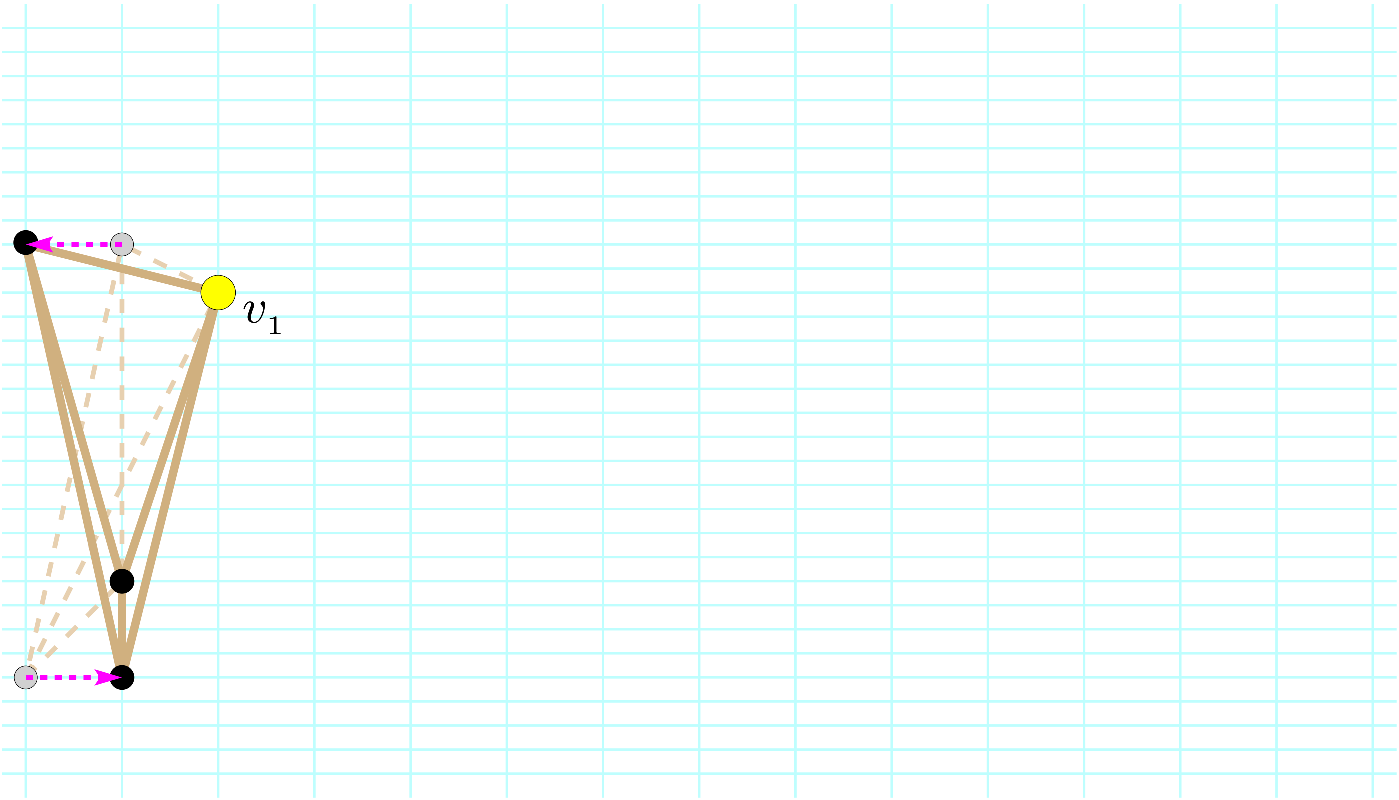

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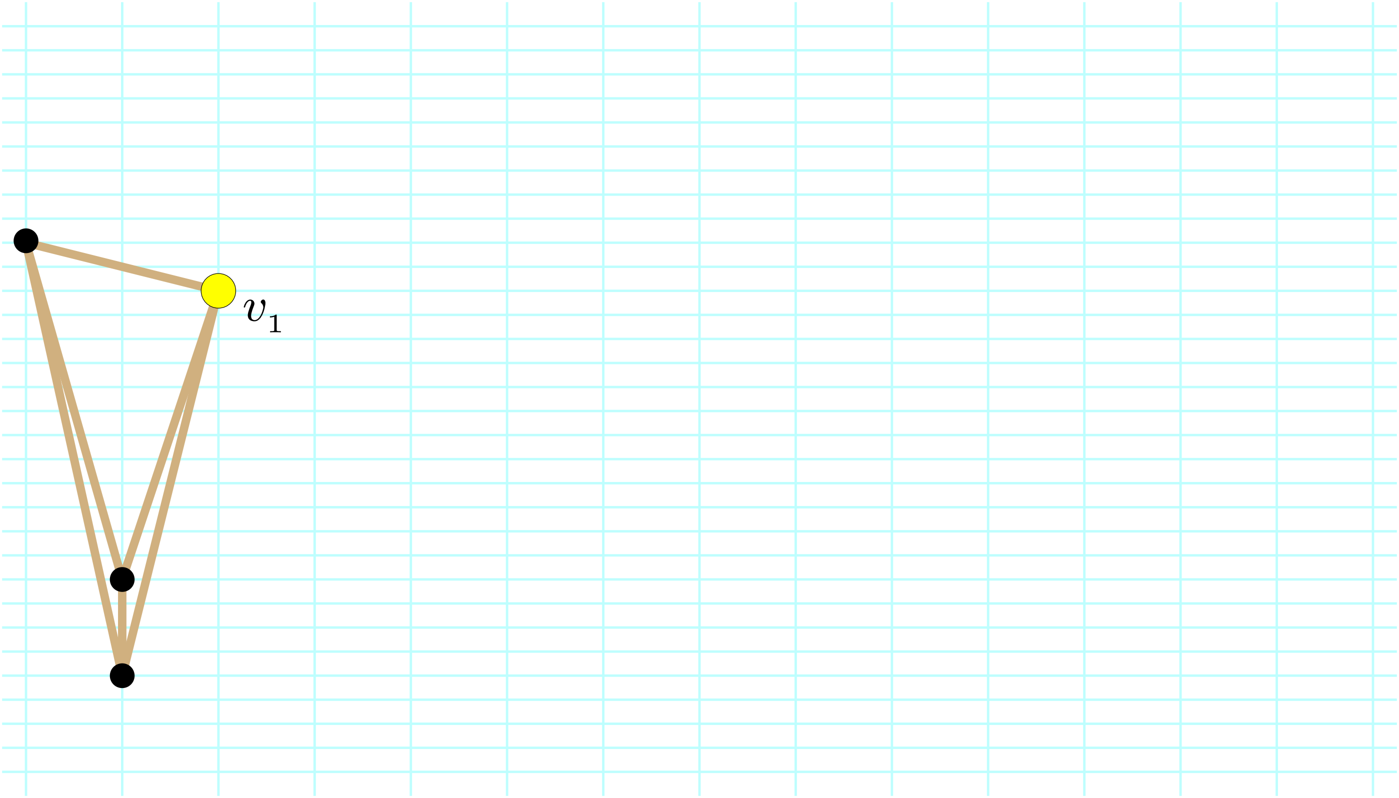

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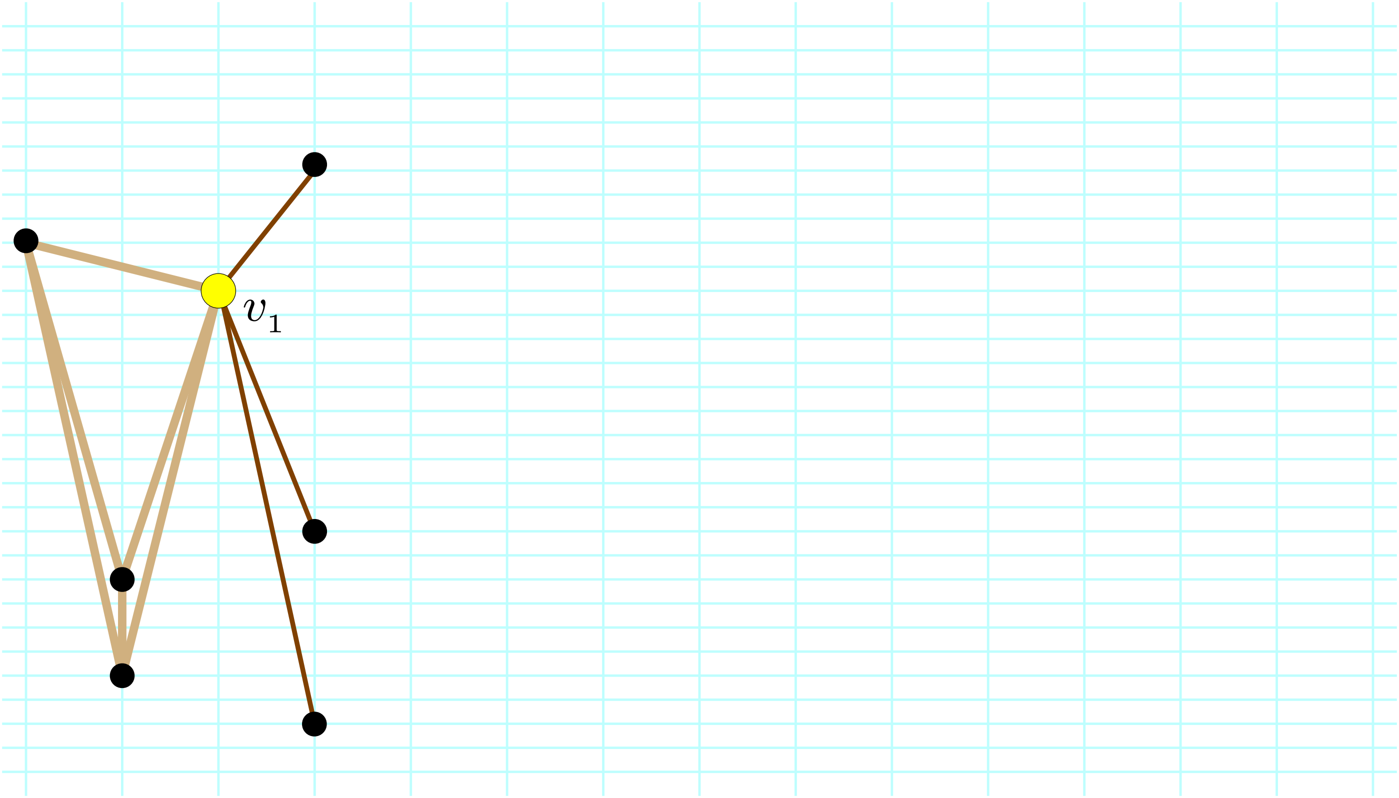

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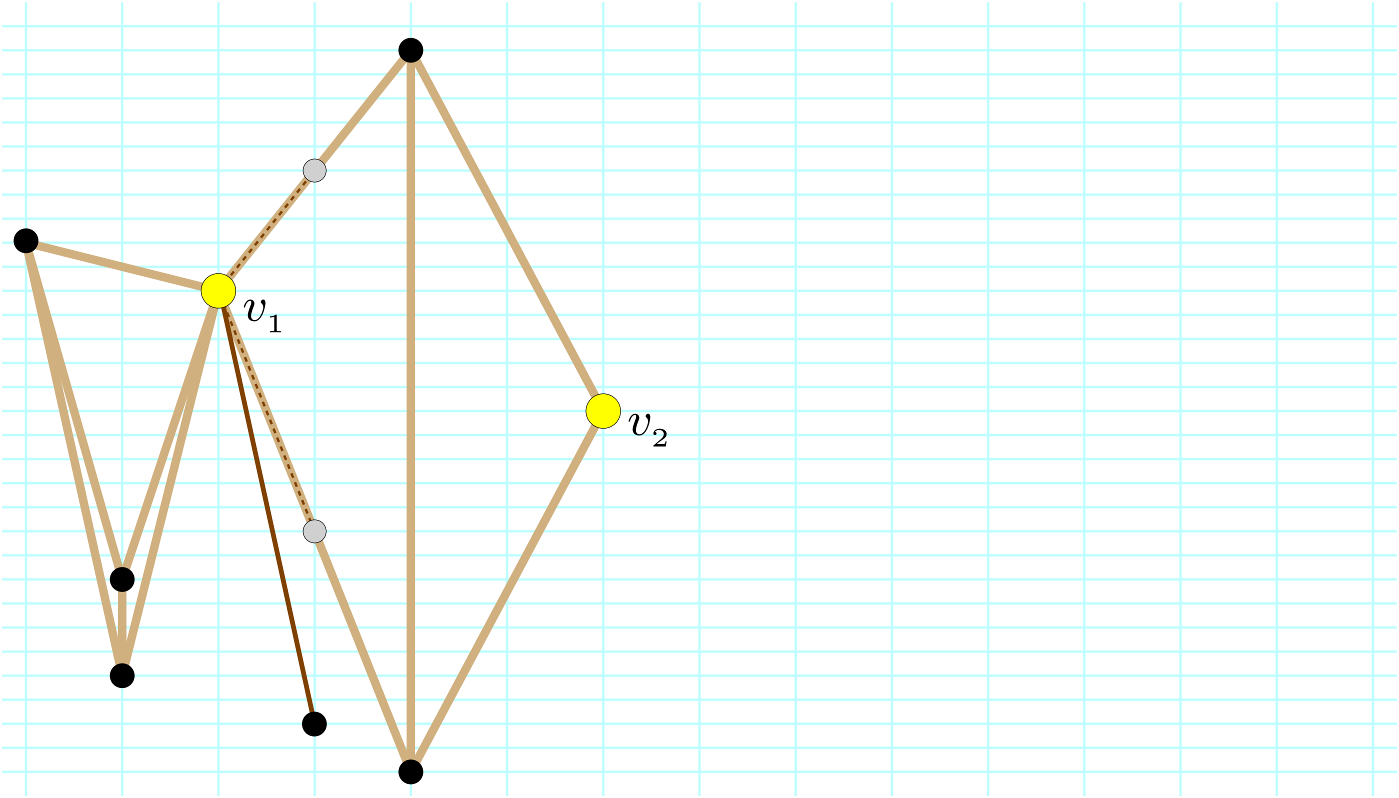

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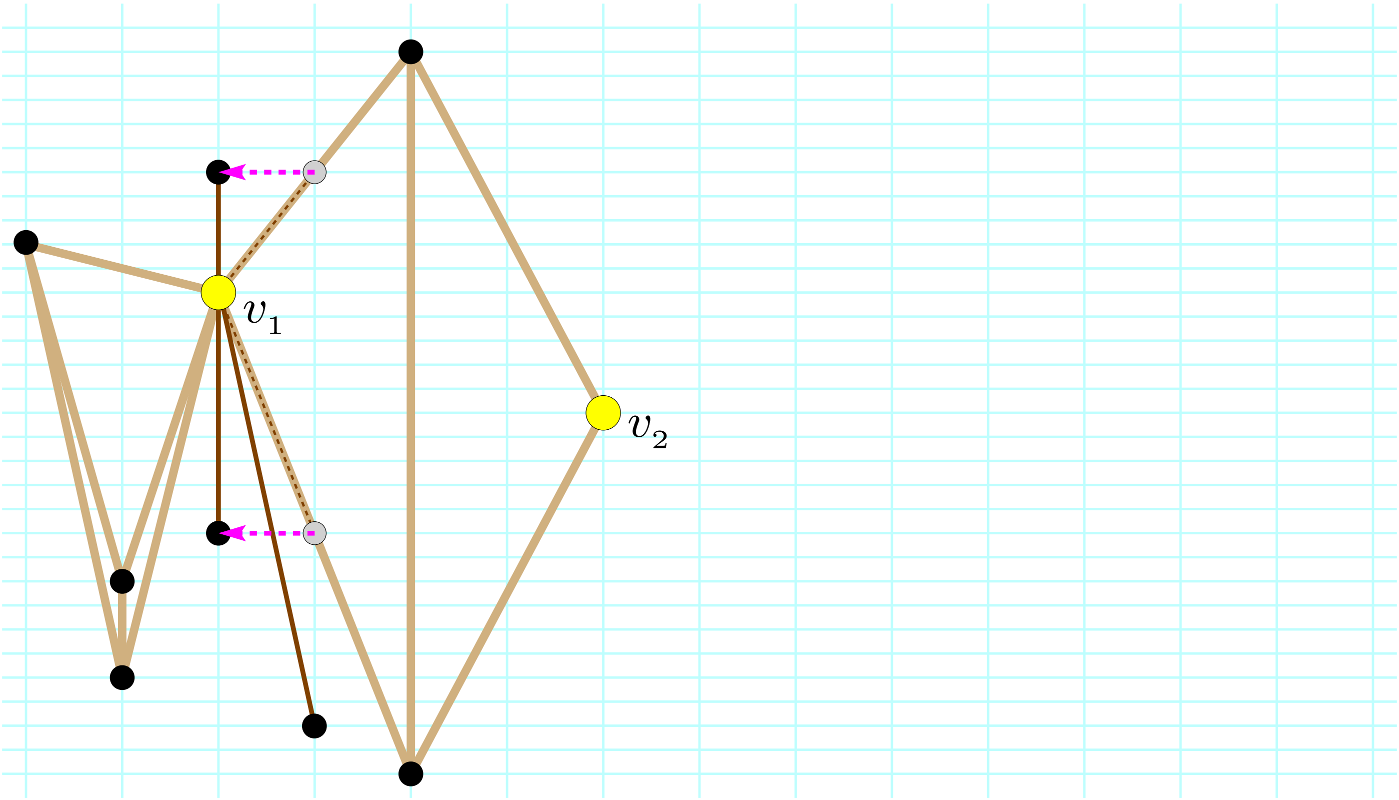

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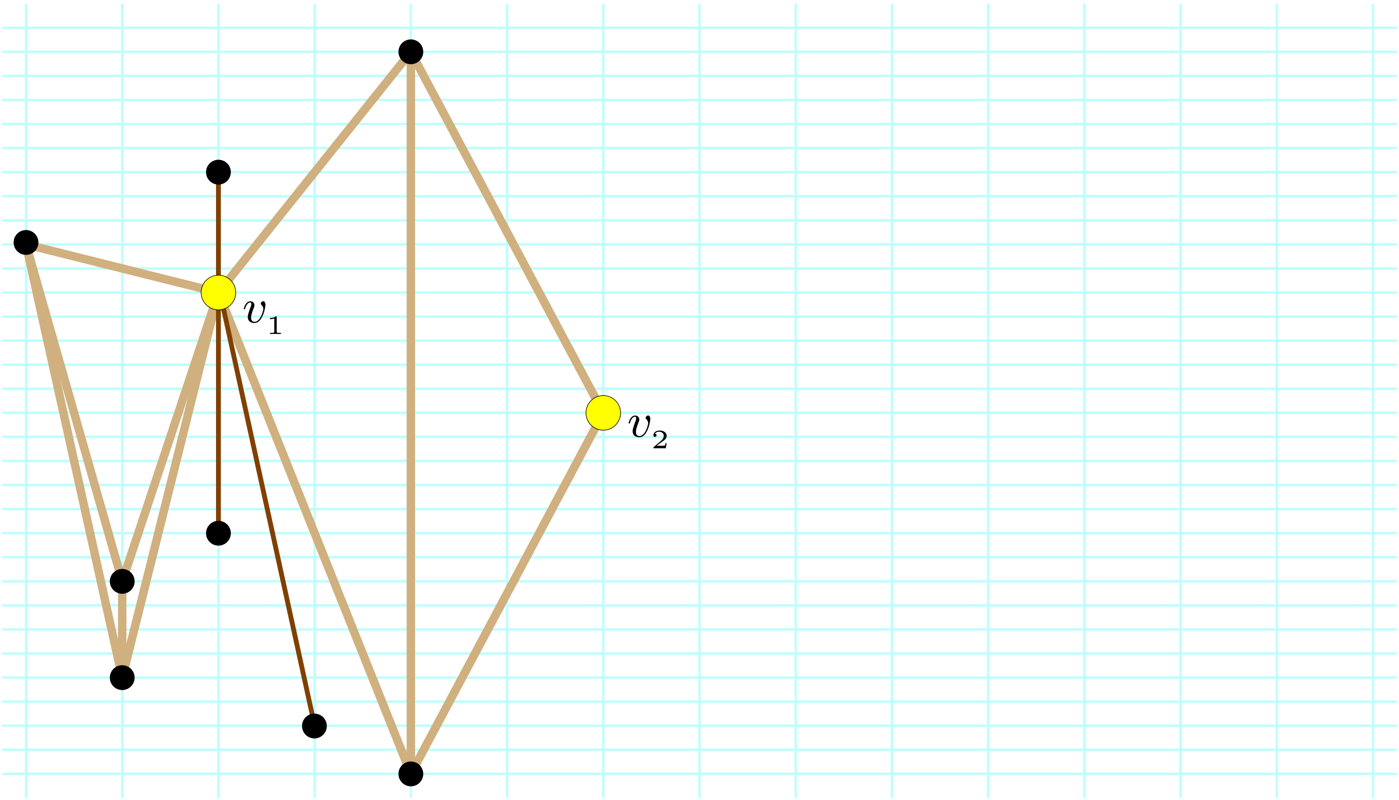

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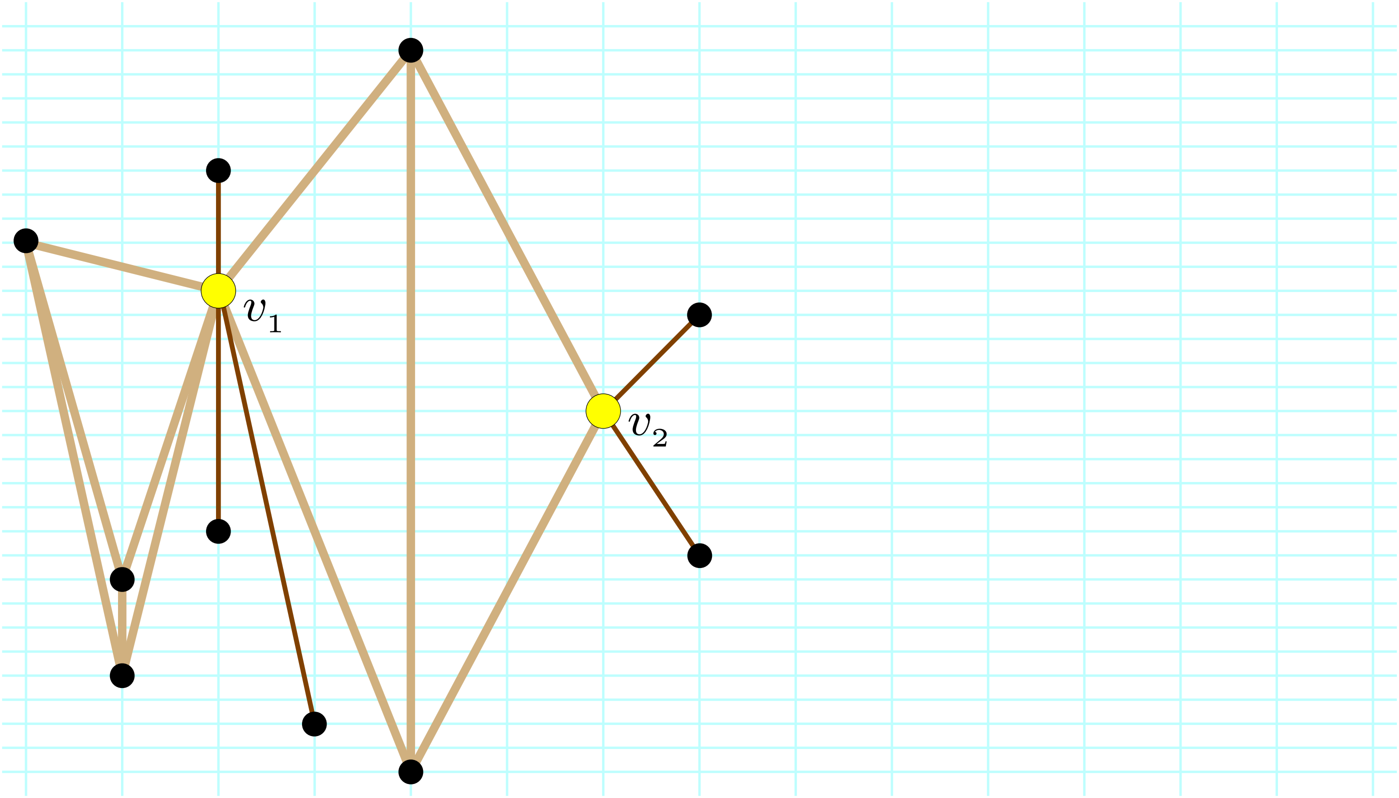

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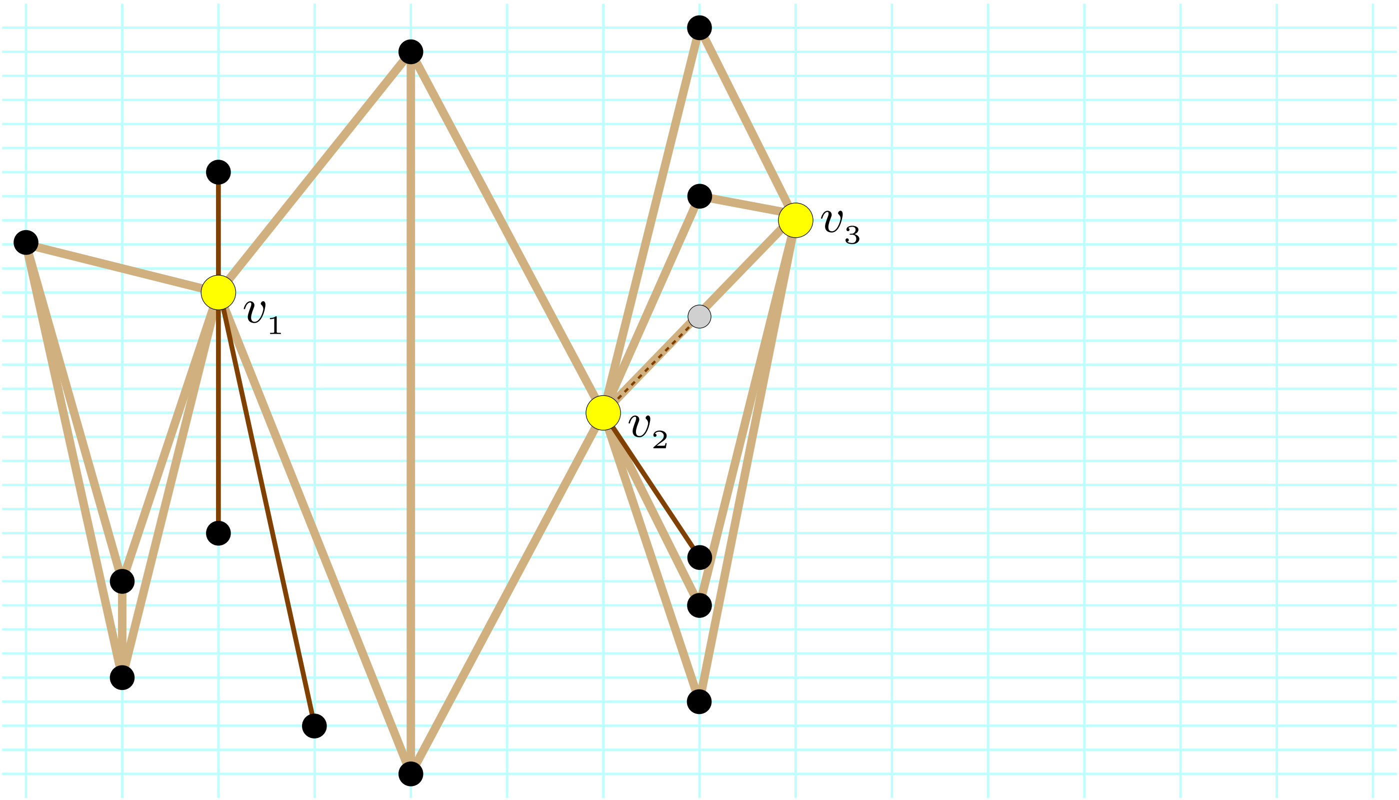

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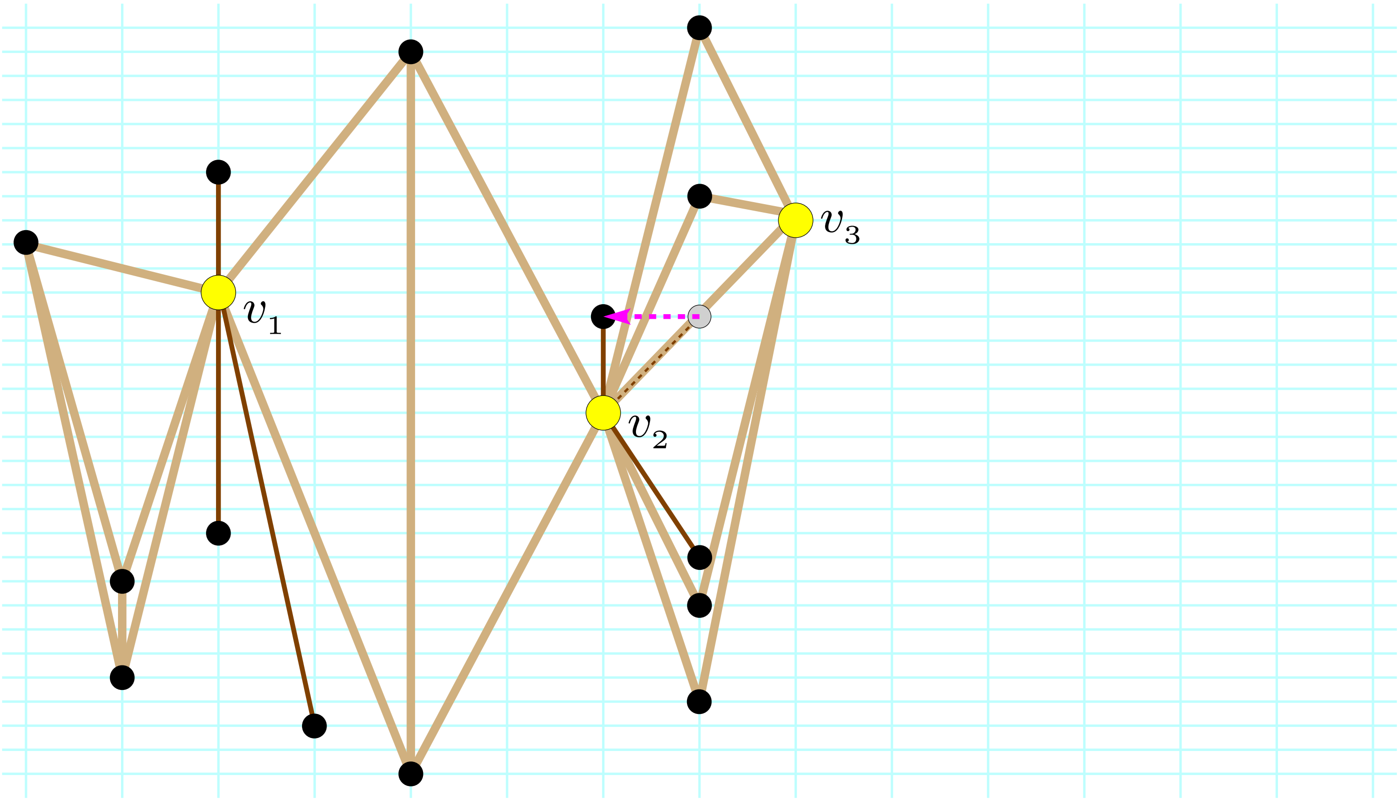

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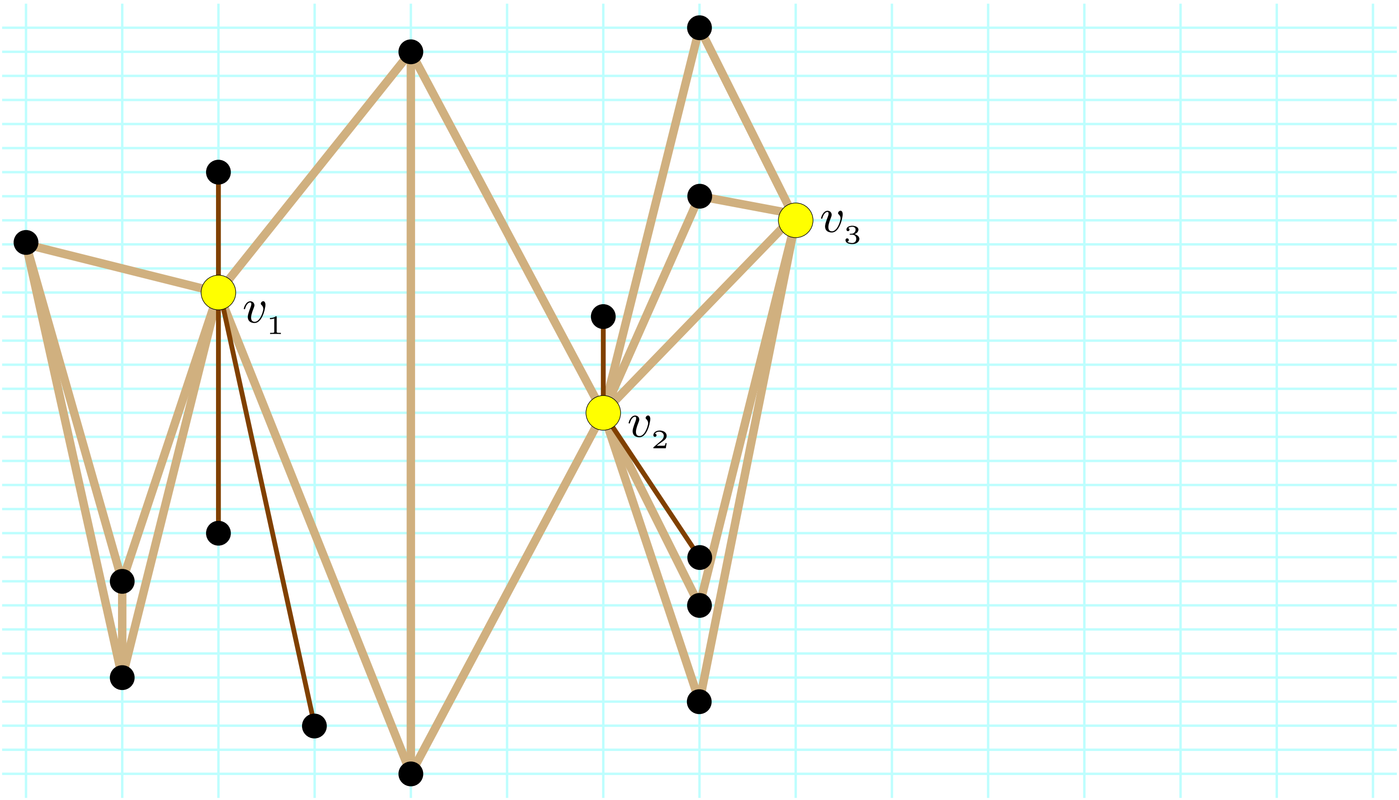

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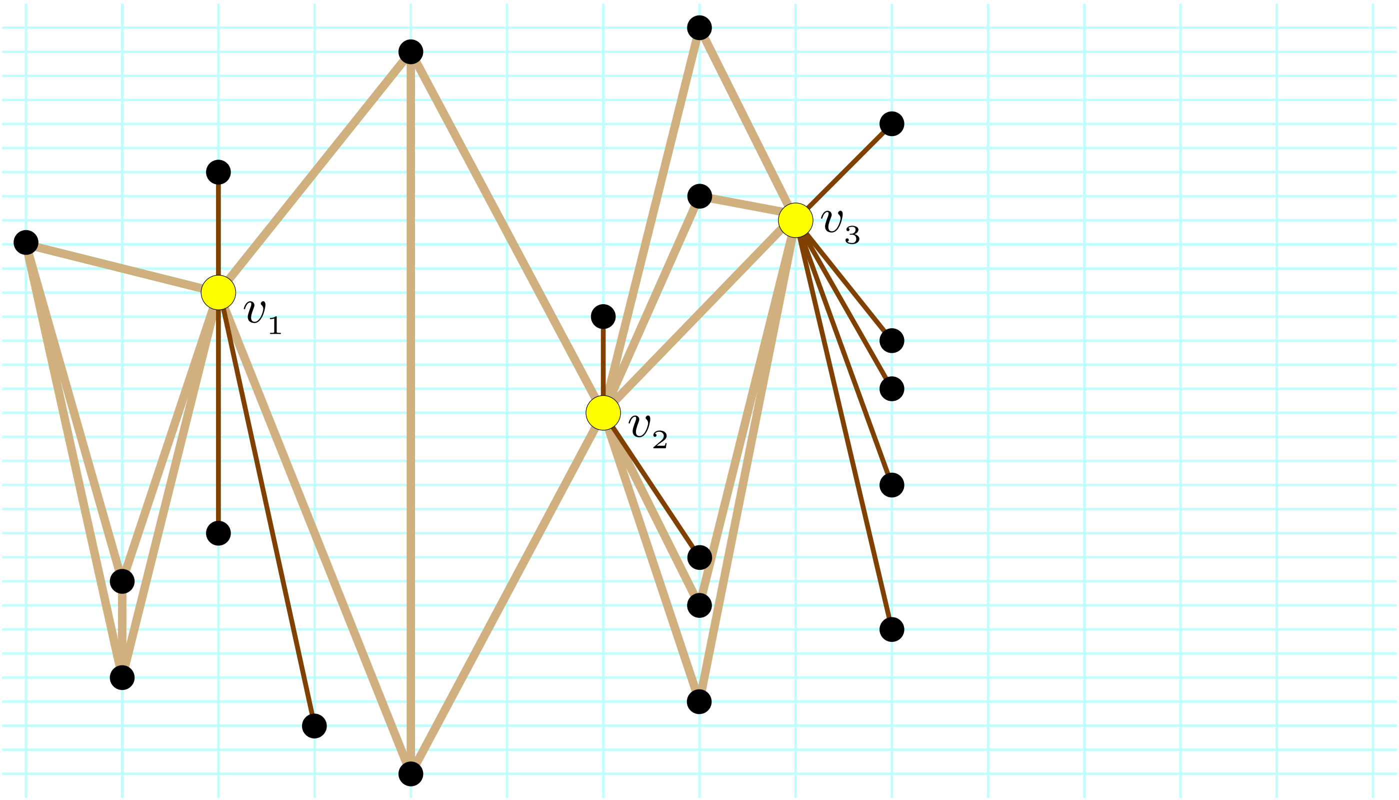

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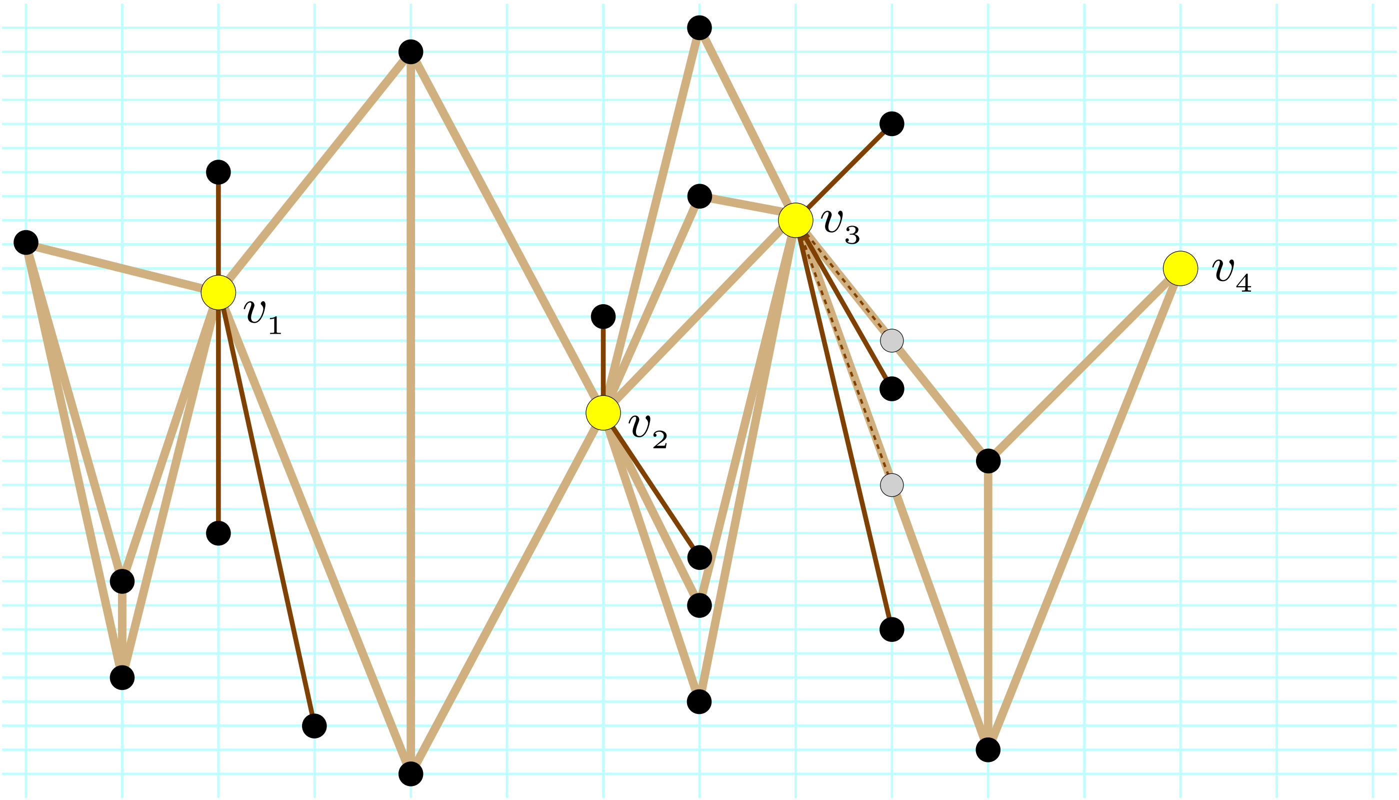

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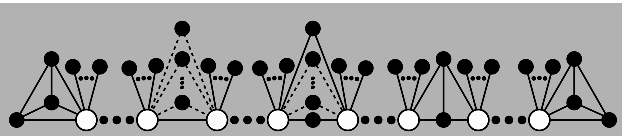

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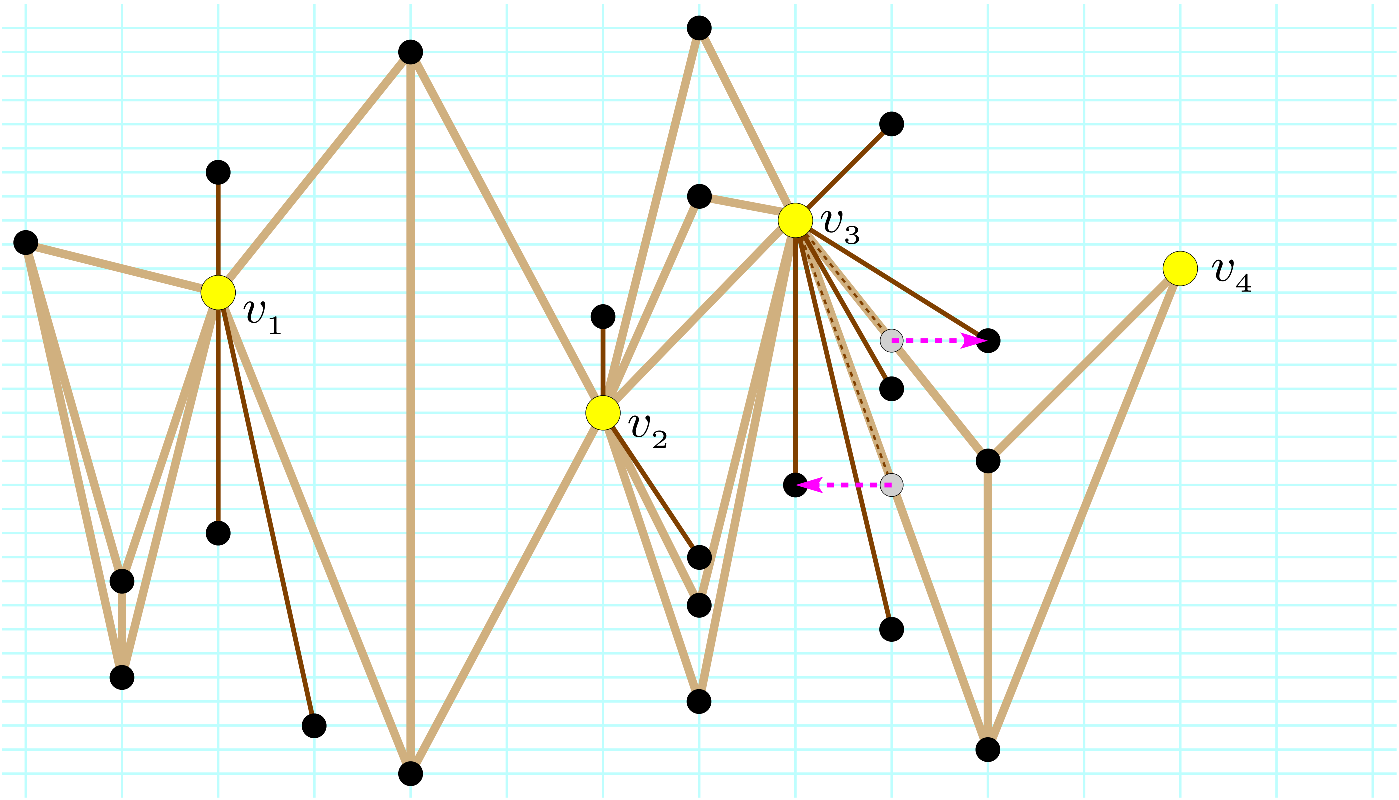

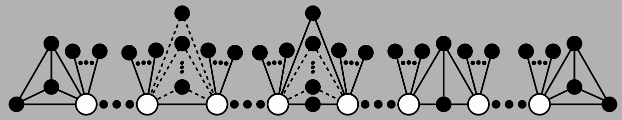

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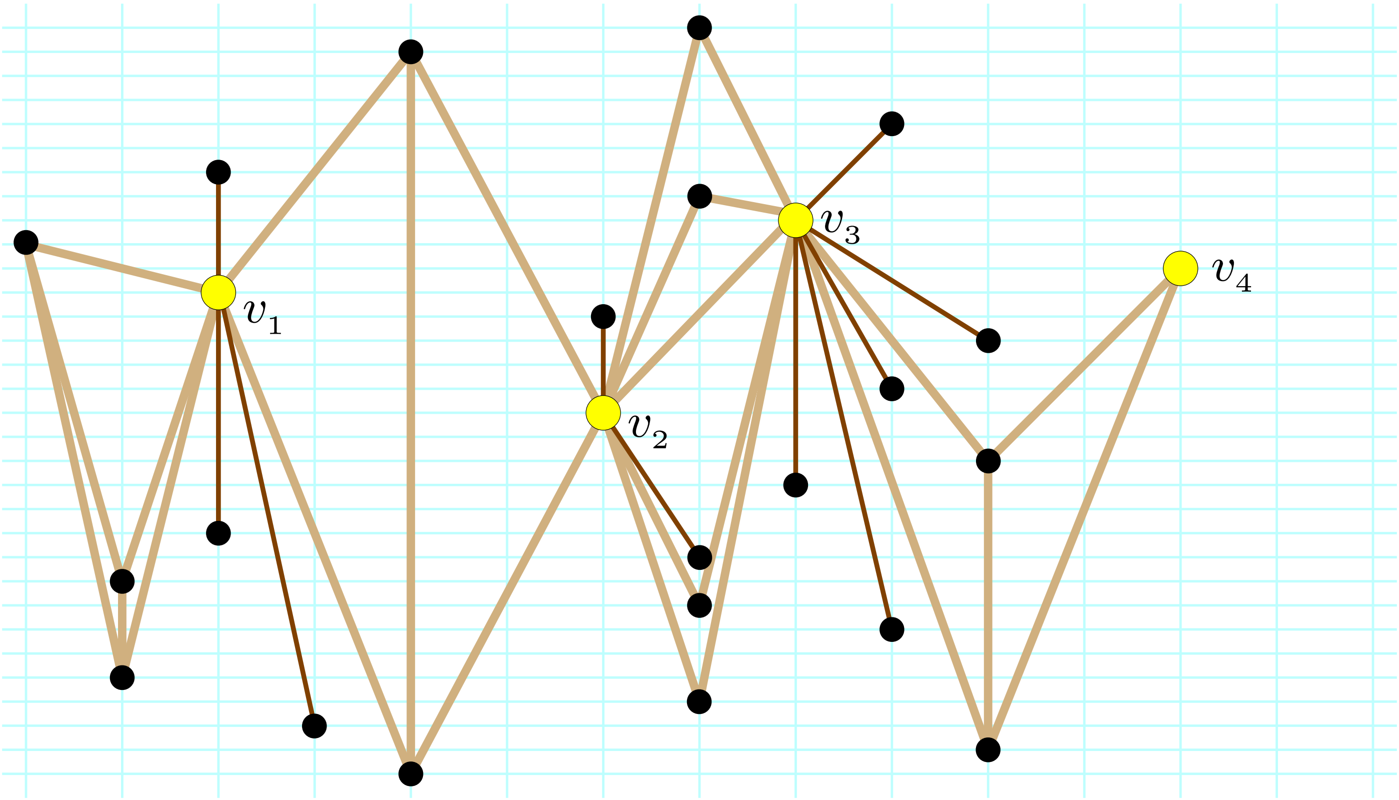

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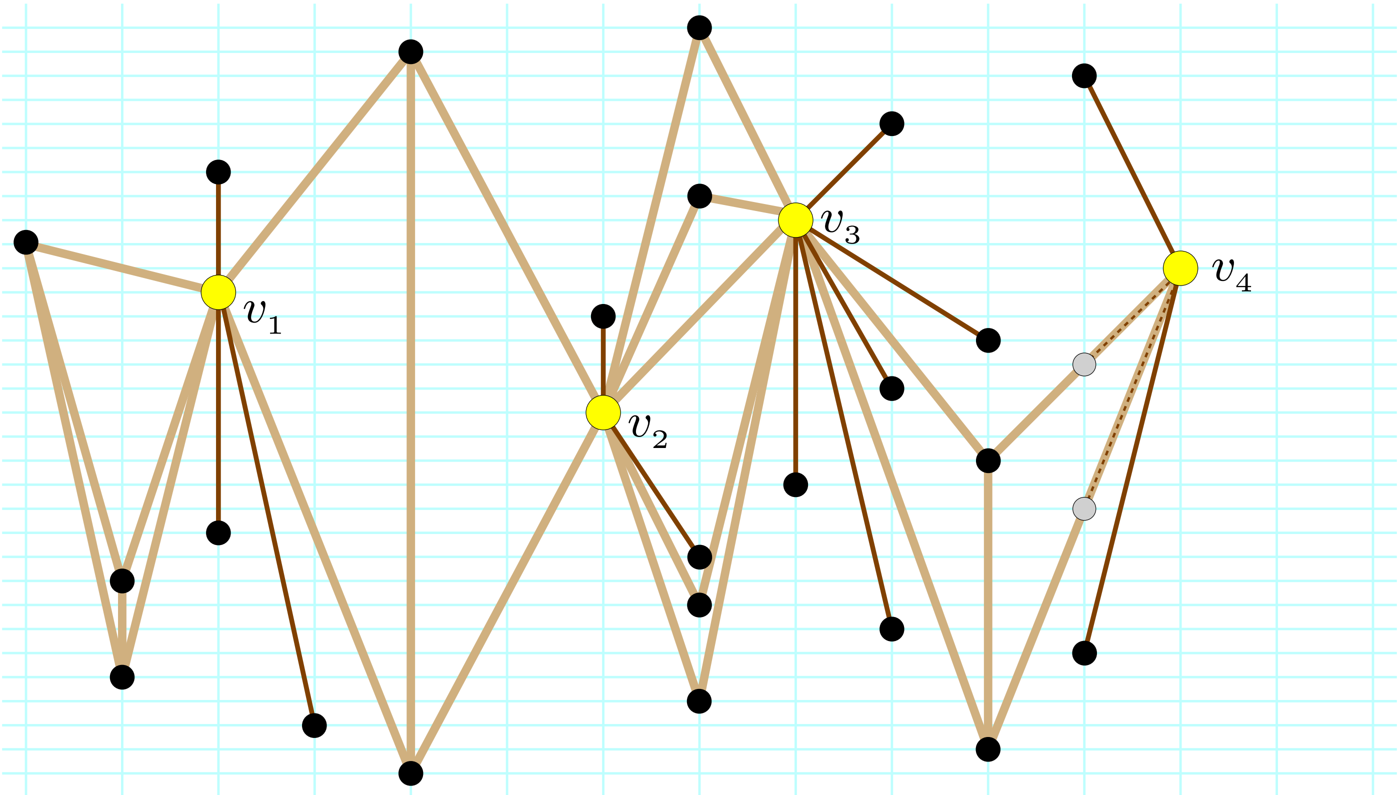

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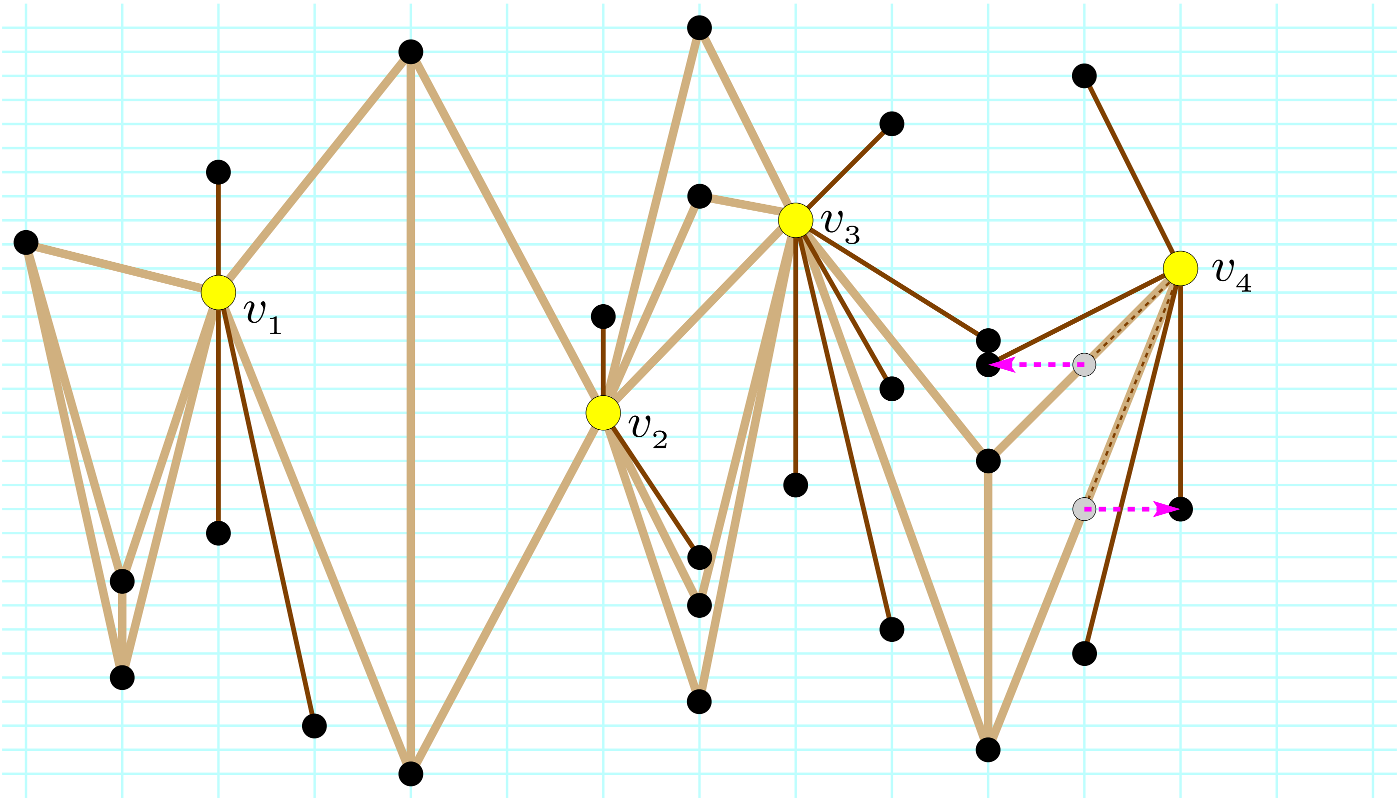

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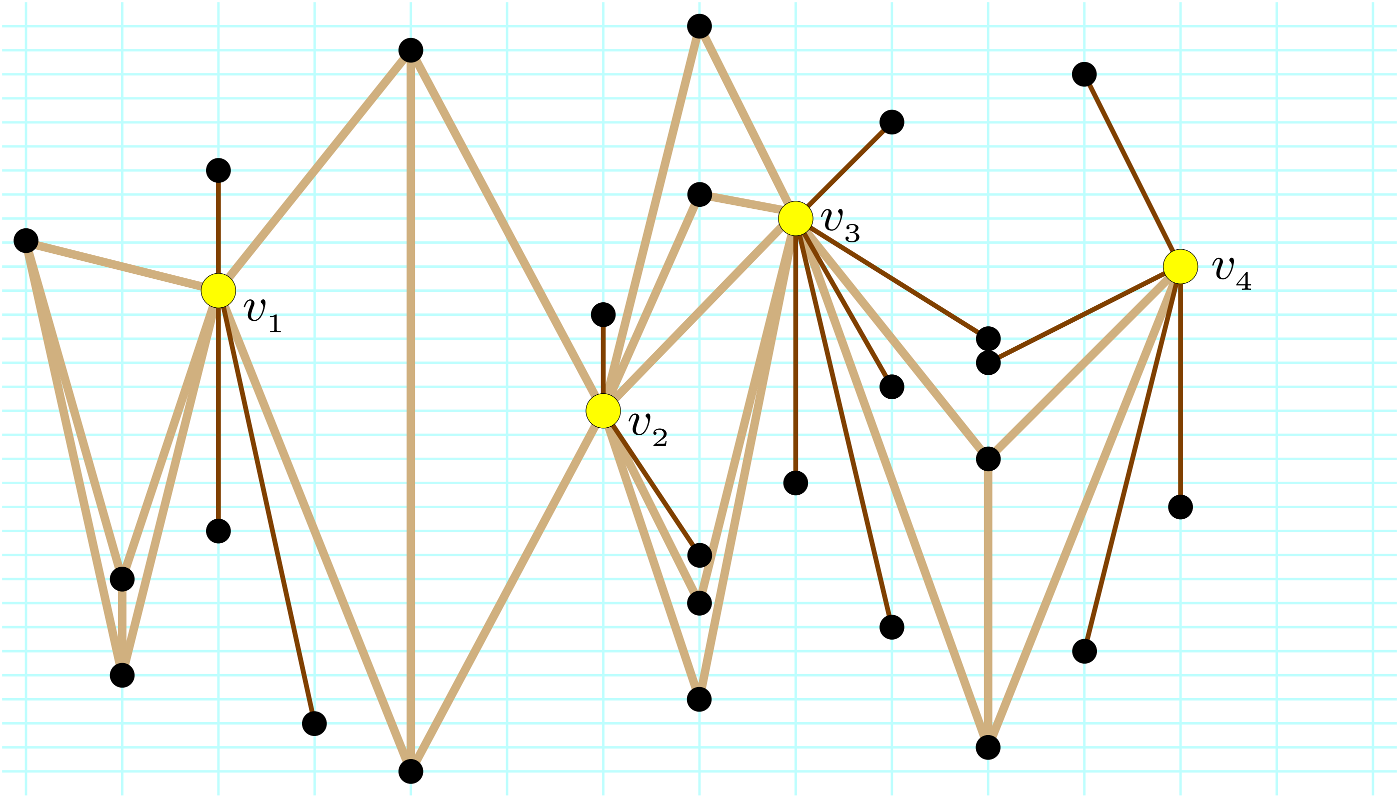

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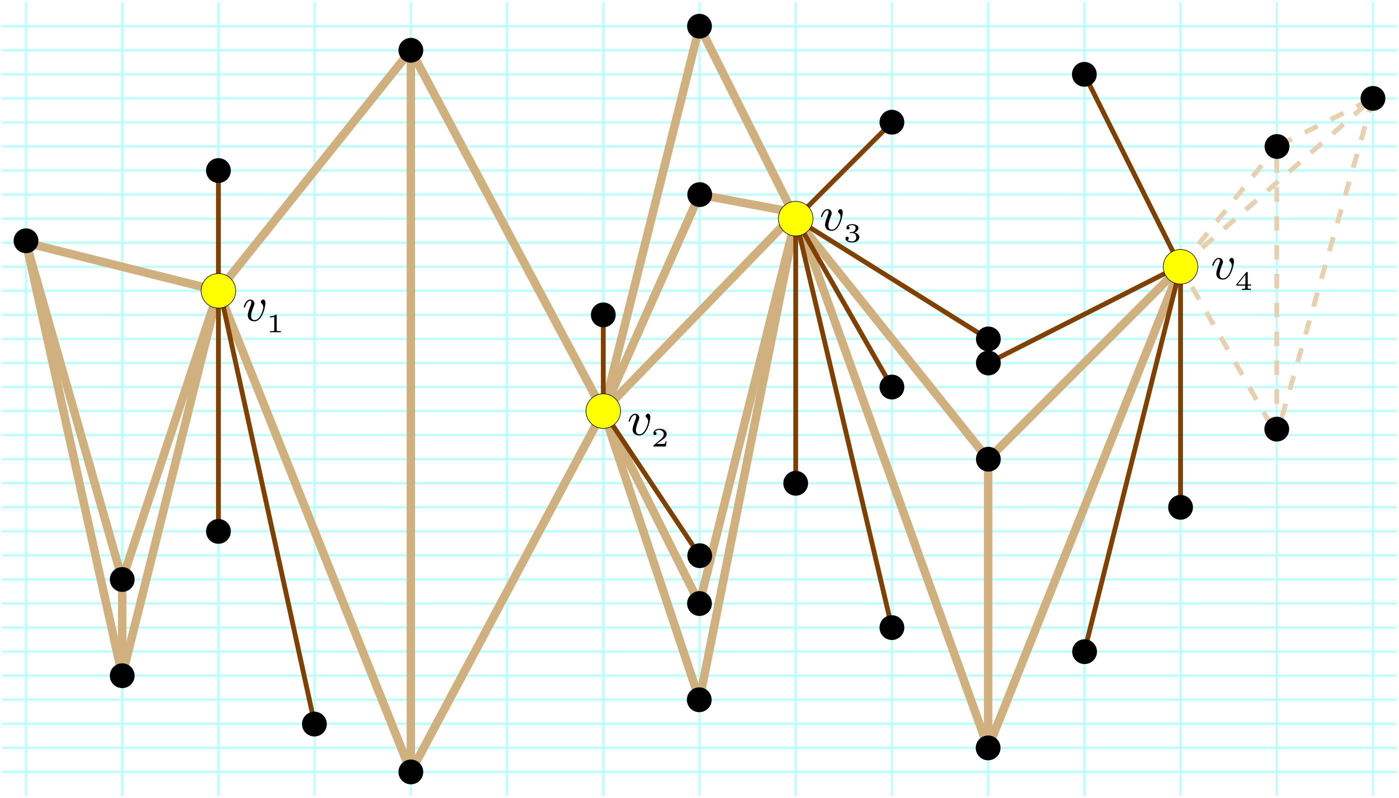

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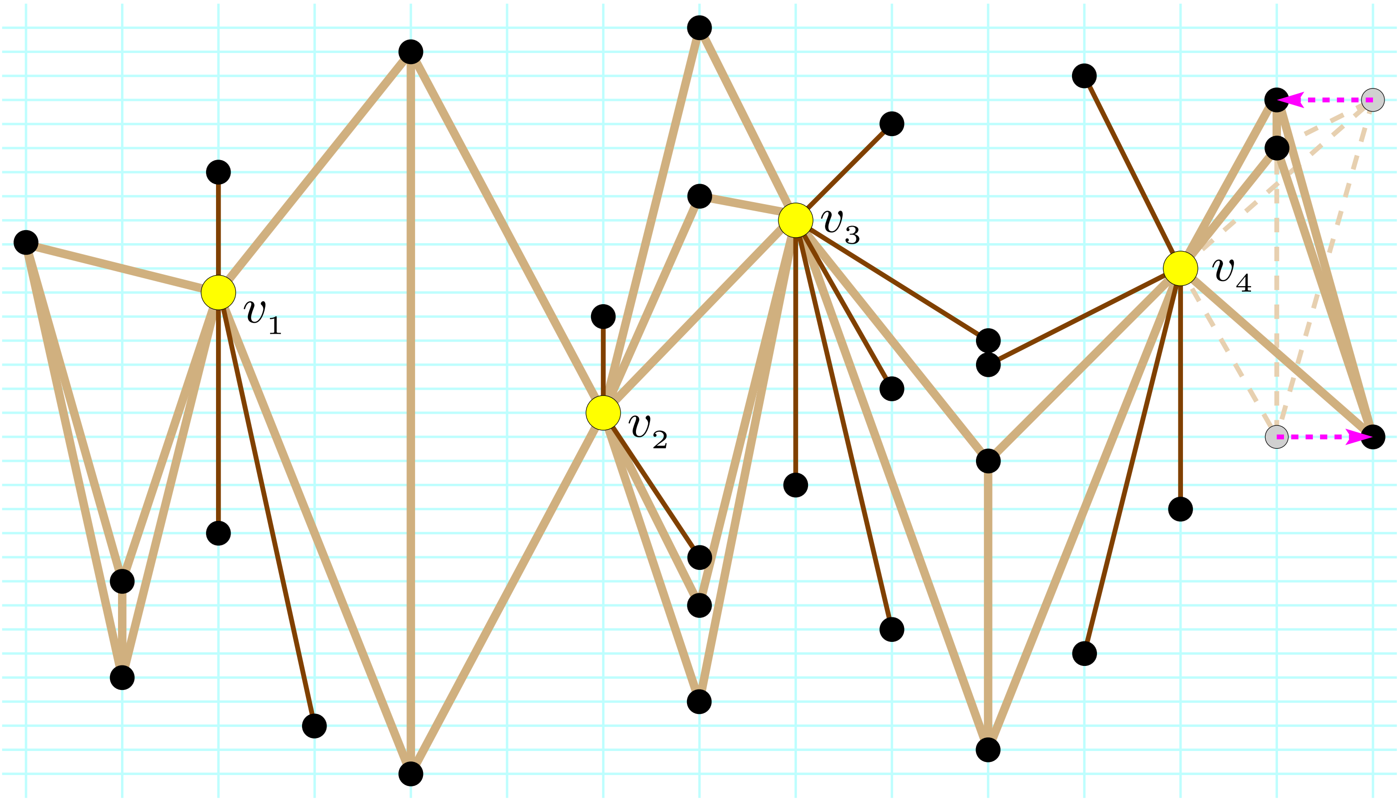

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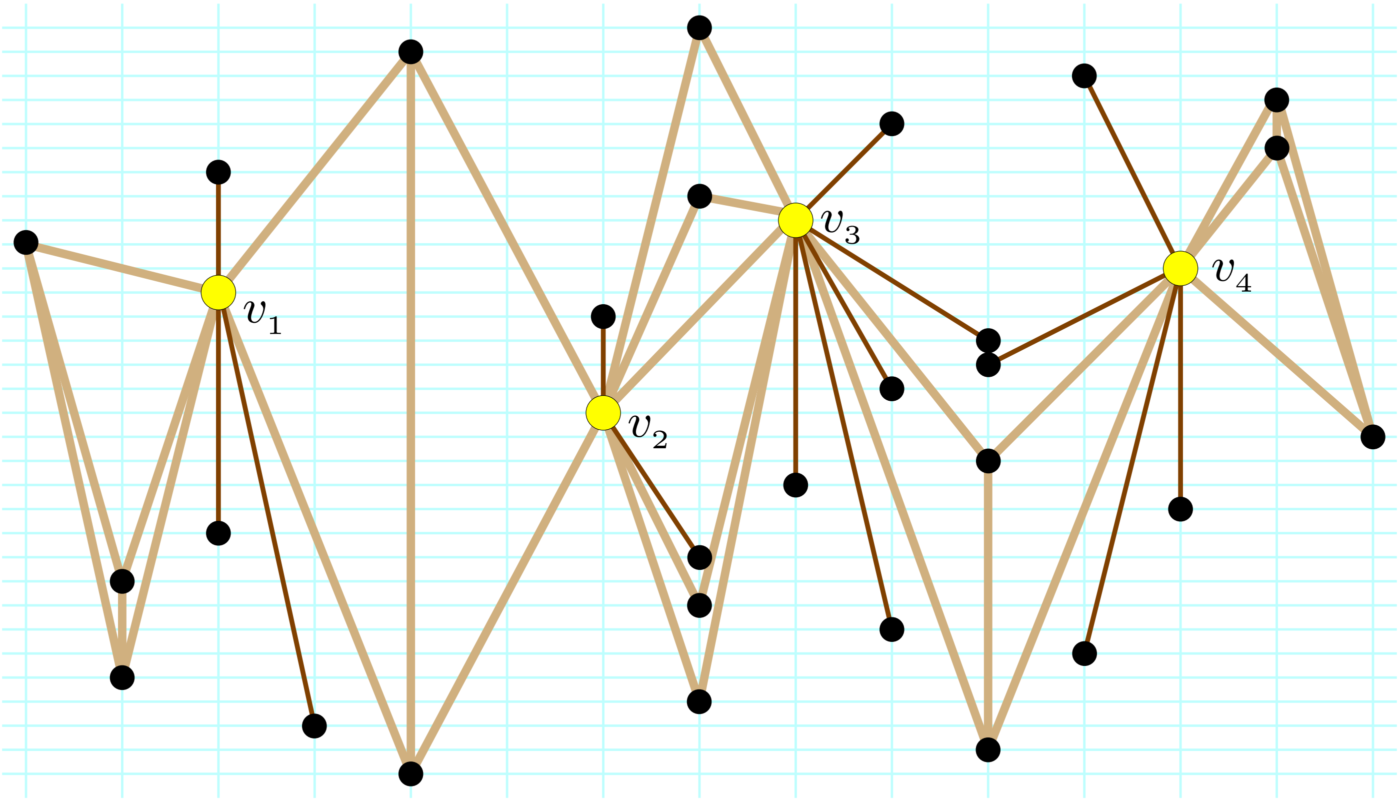
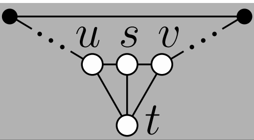


2-Connected Extended 3-Spider



- Drawing a 2-connected extended degree-3 spider:

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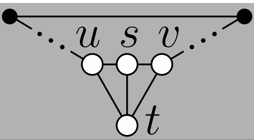

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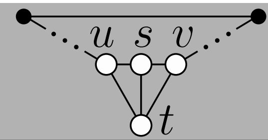

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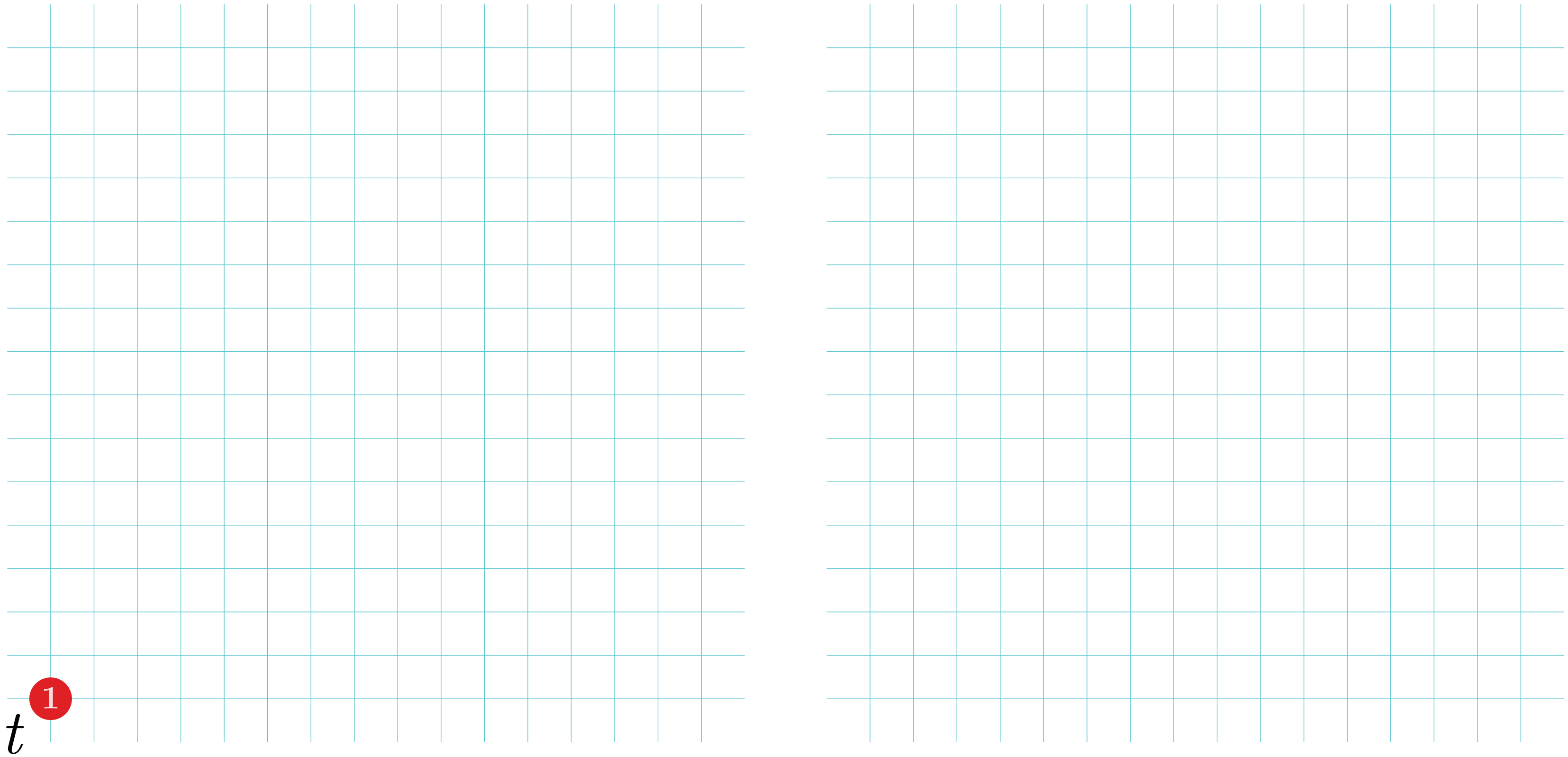

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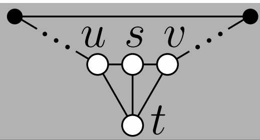

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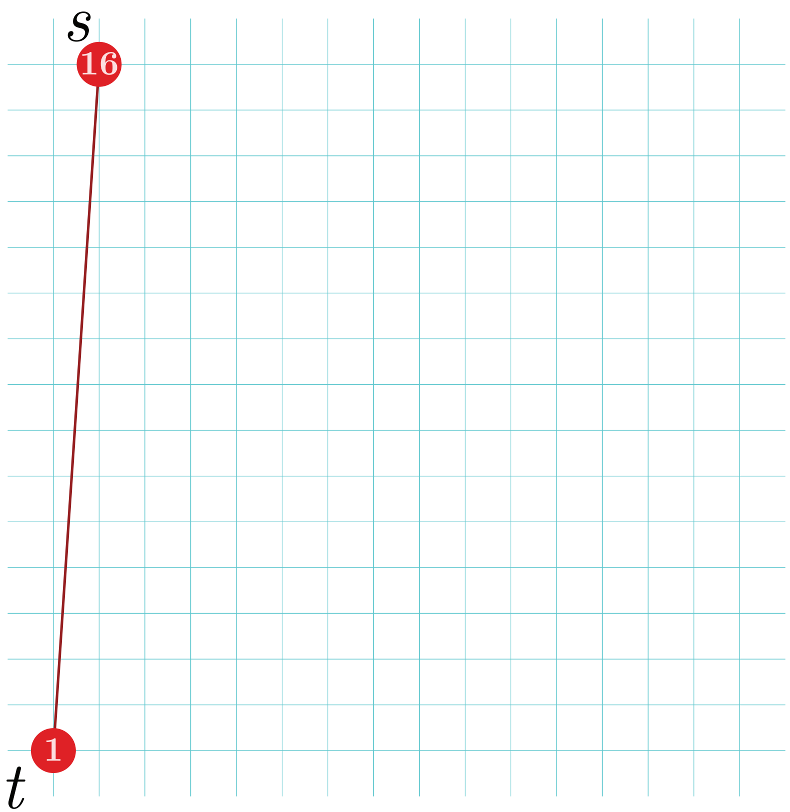

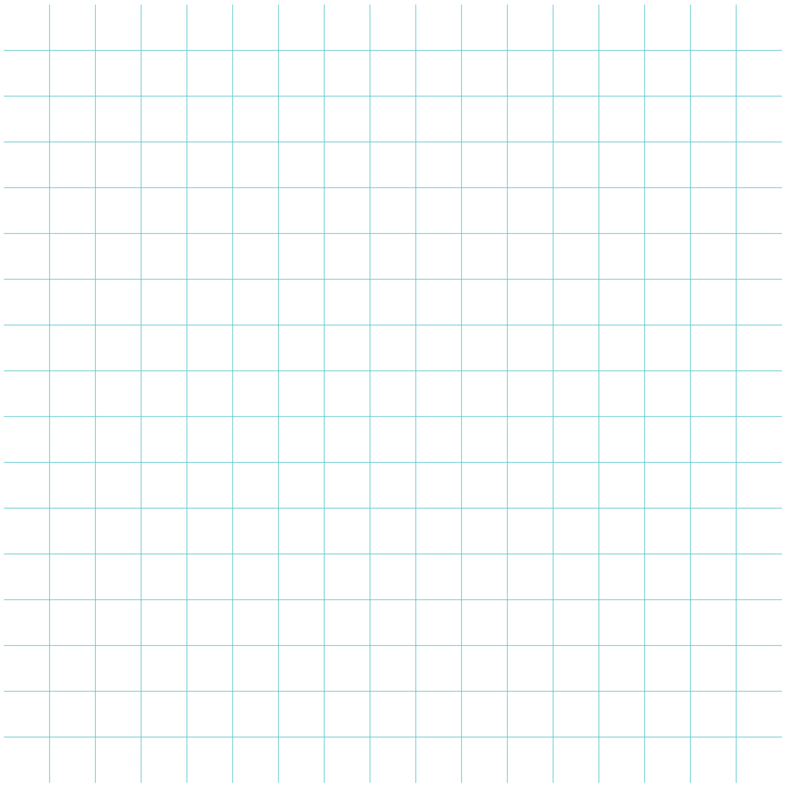

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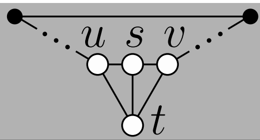

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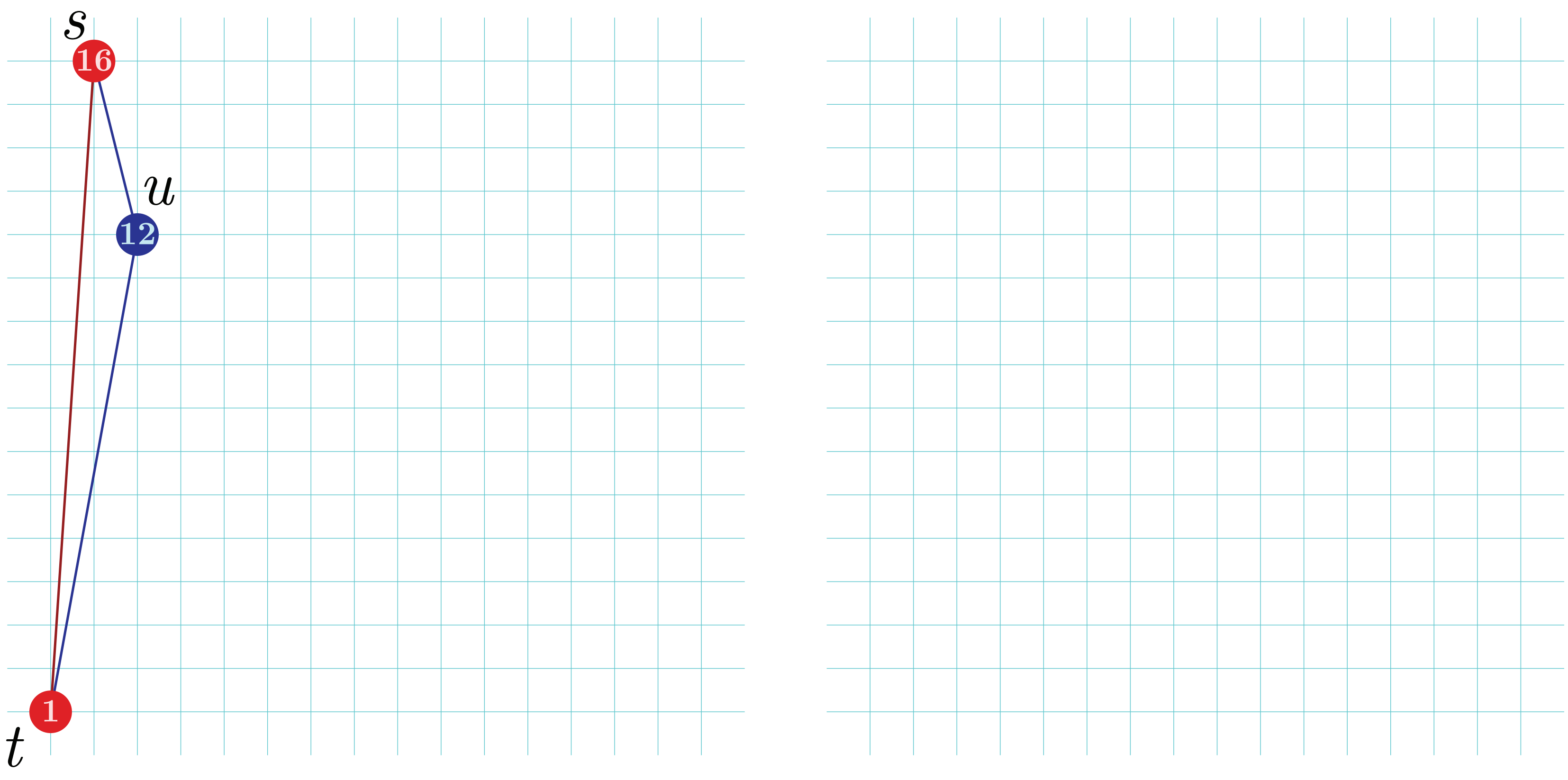

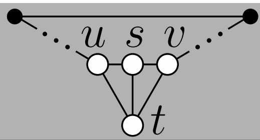

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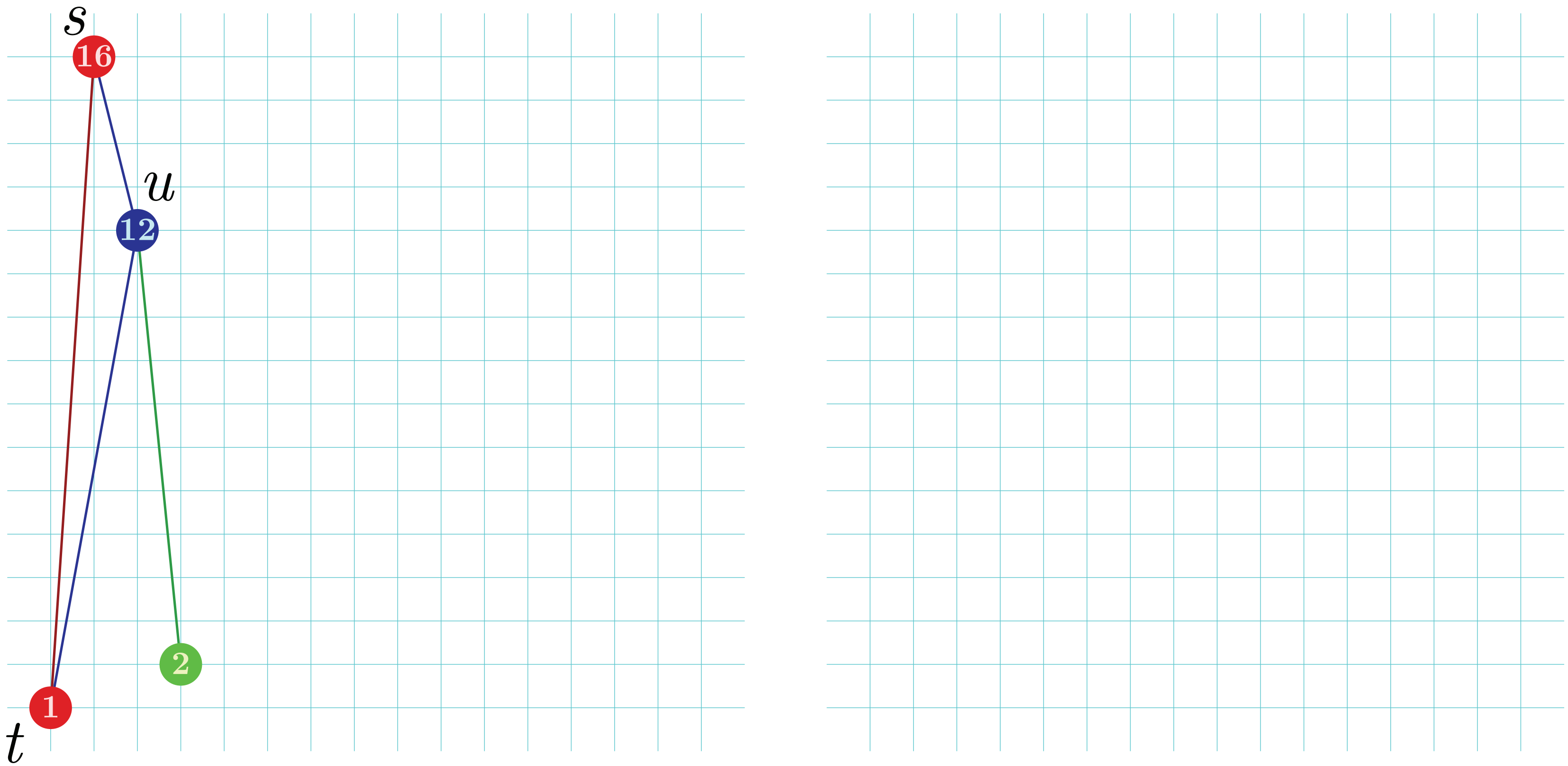

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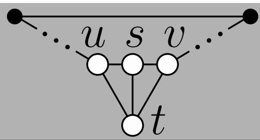

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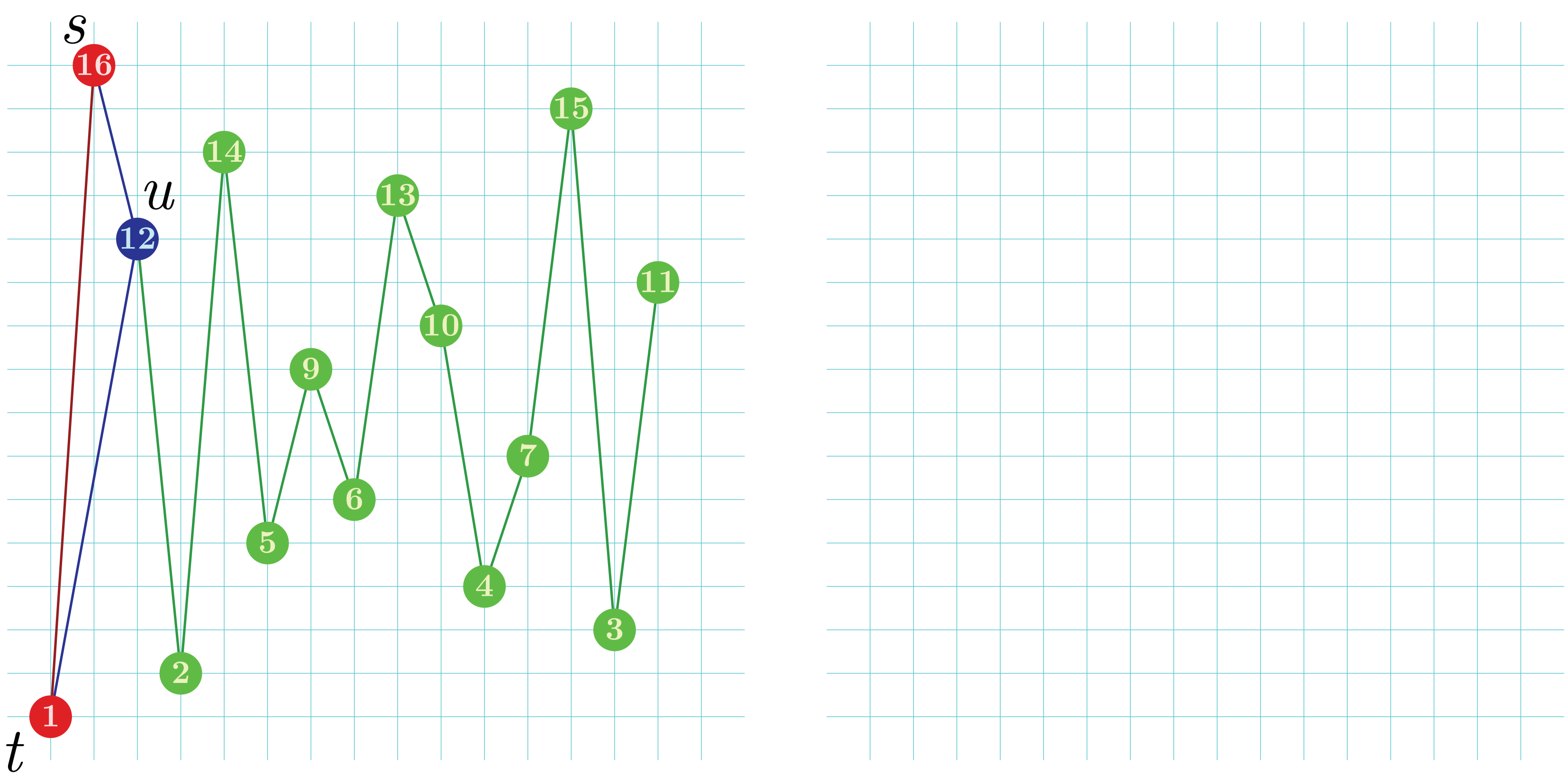

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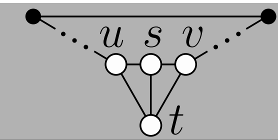

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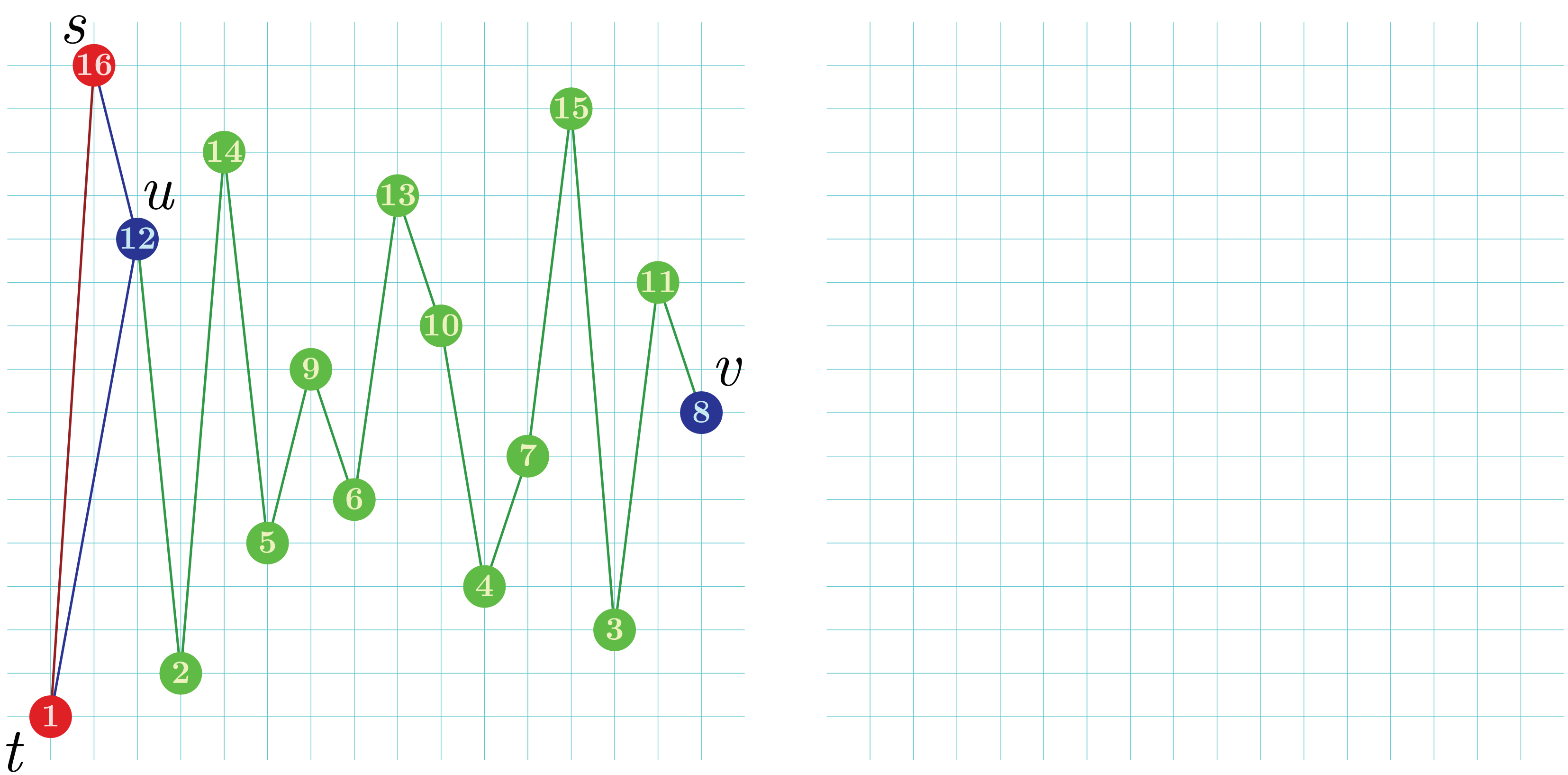

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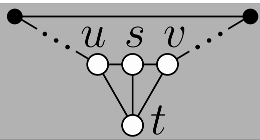

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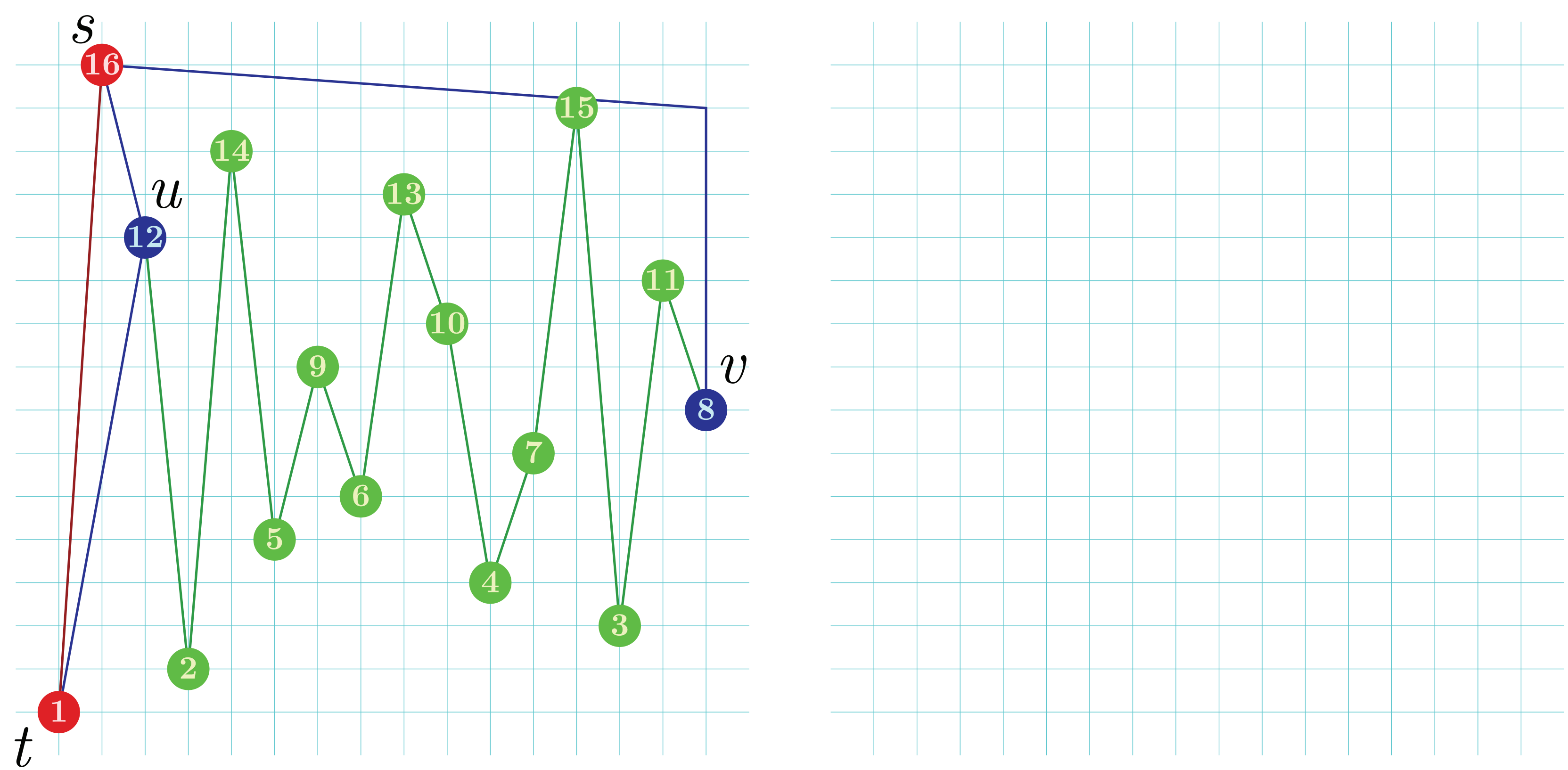

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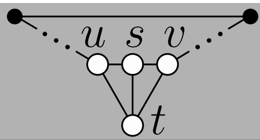

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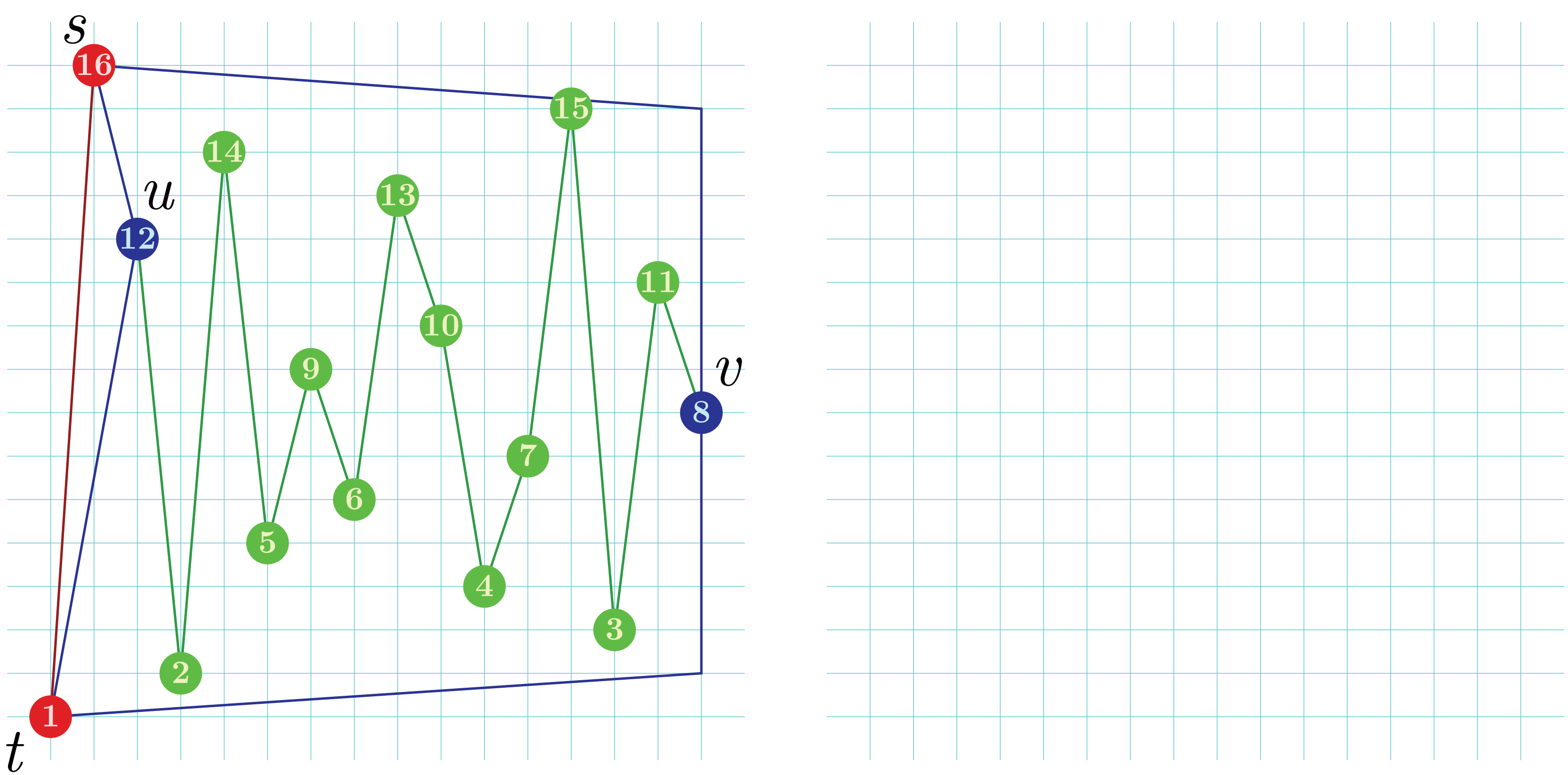

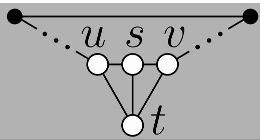

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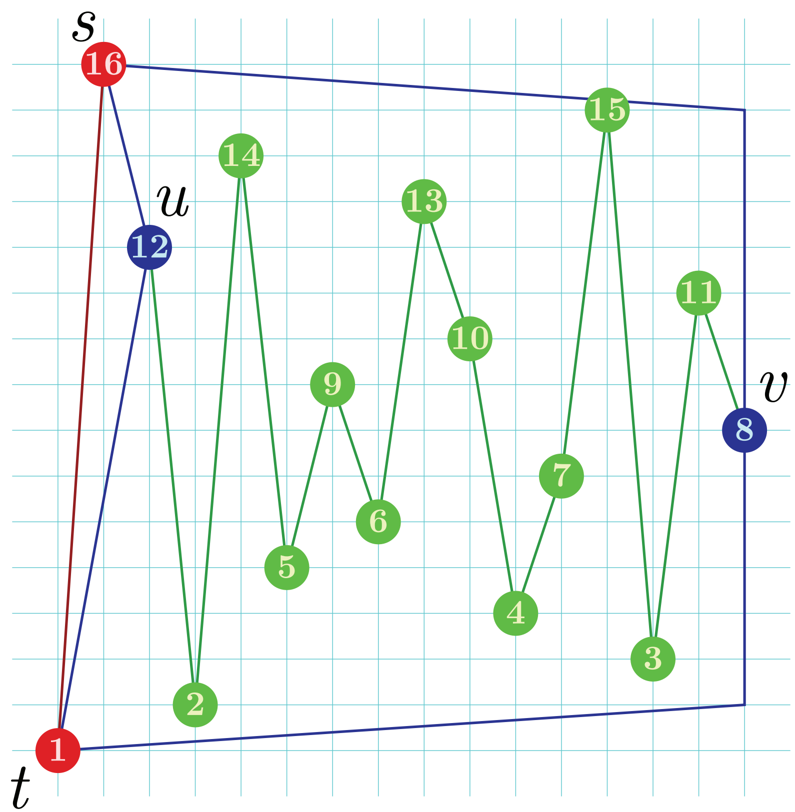

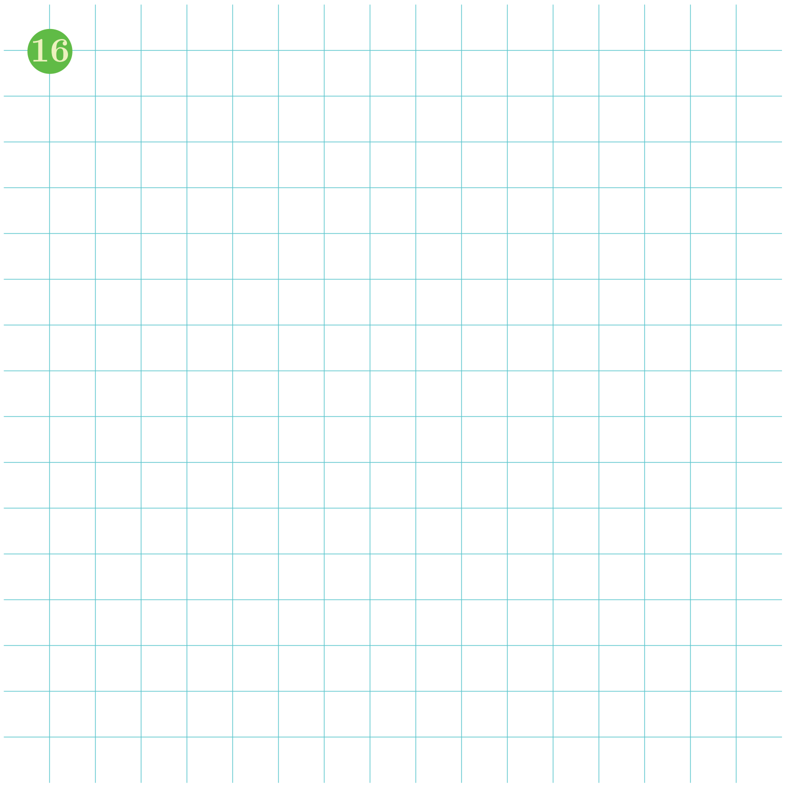

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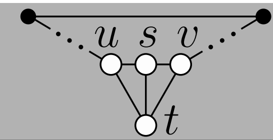

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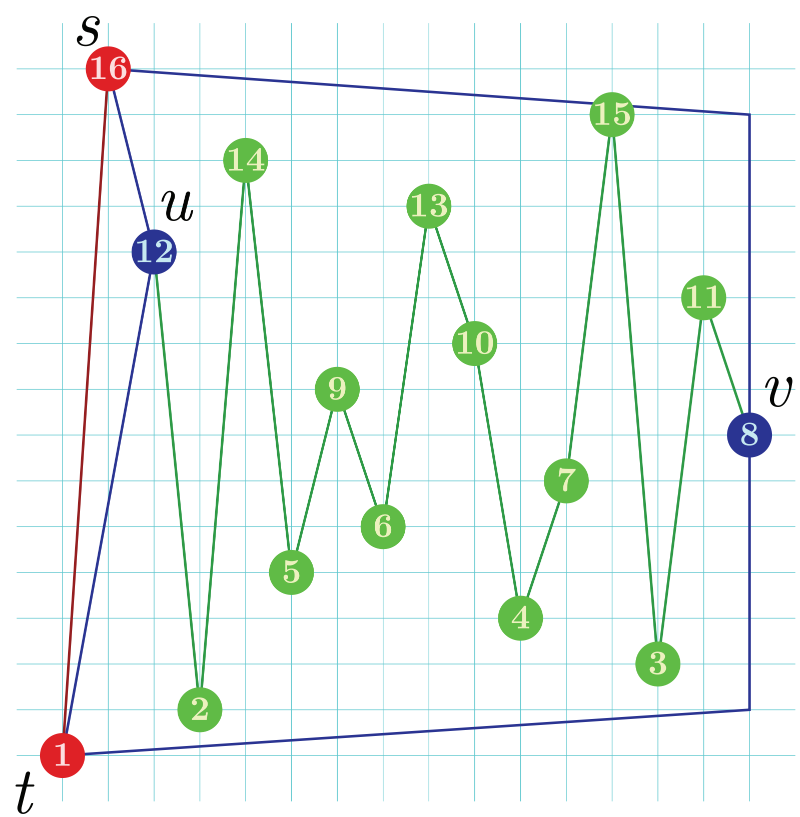

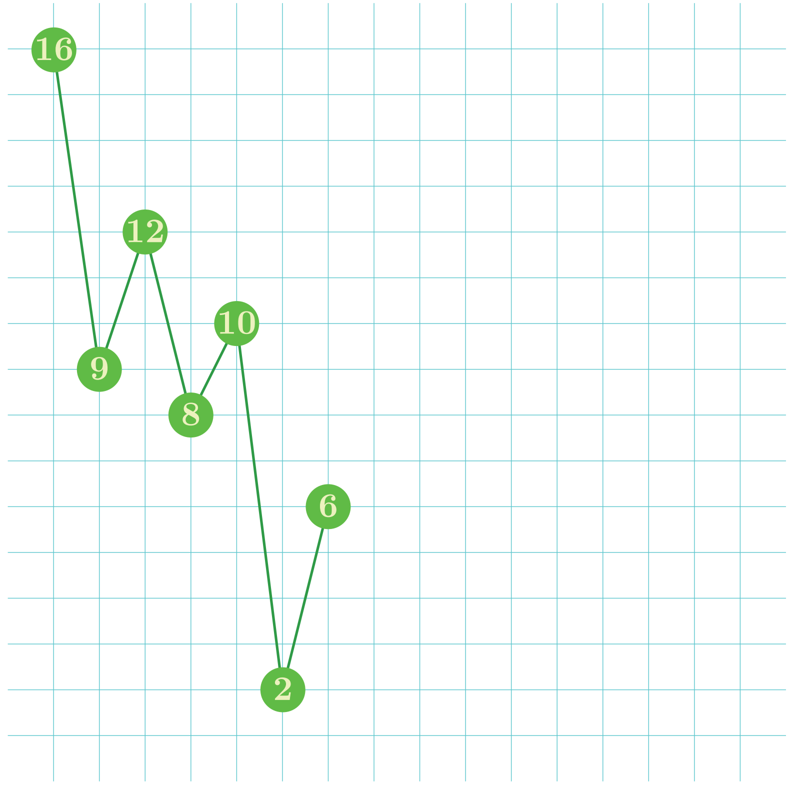

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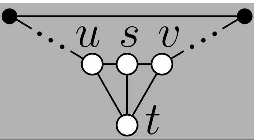

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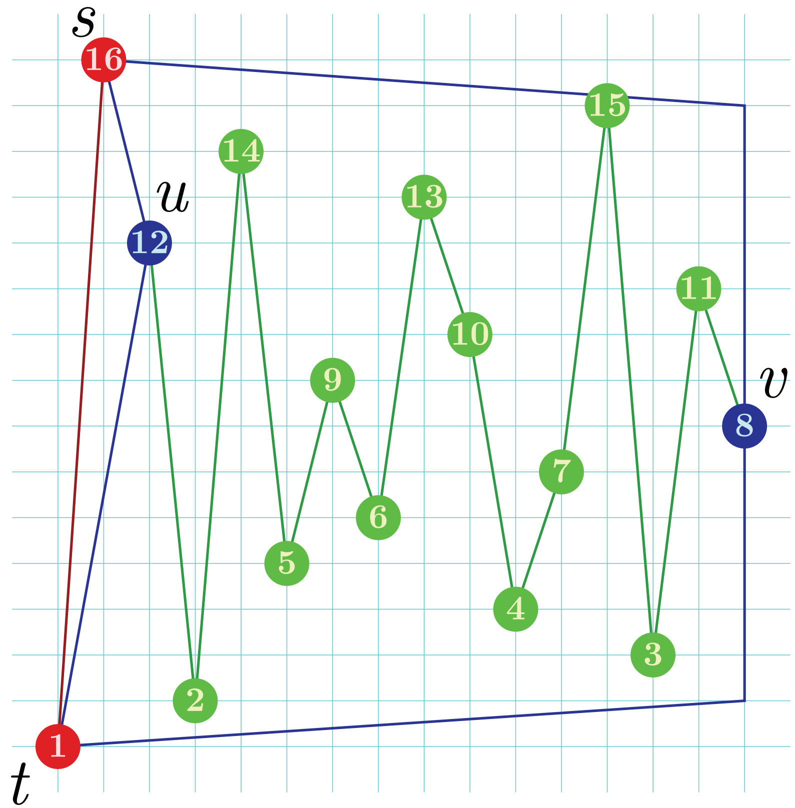

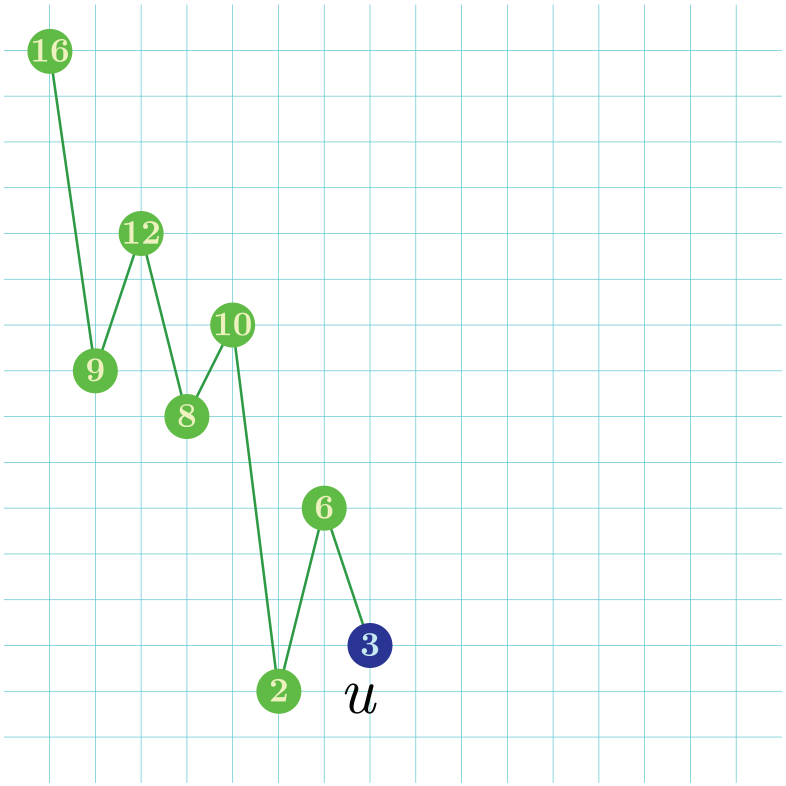

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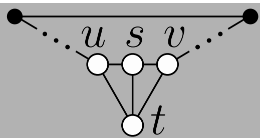

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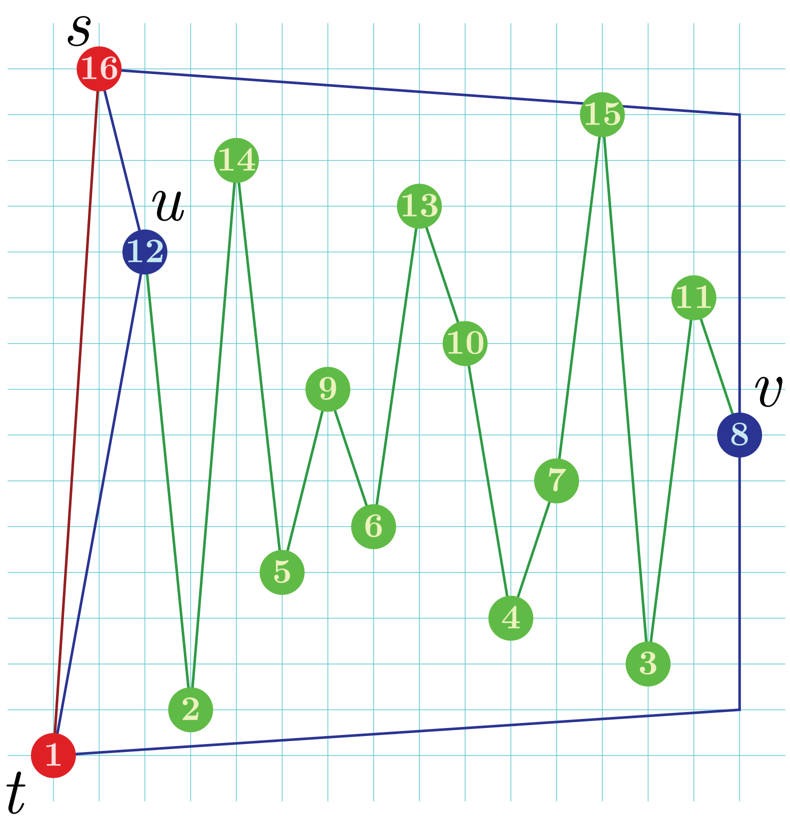
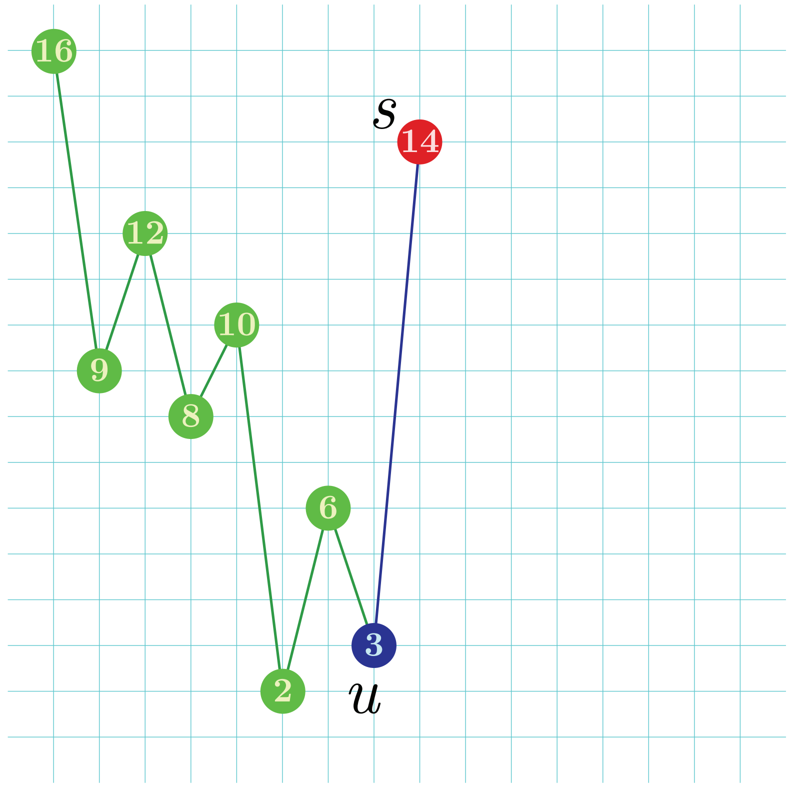
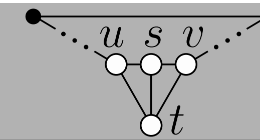




2-Connected Extended 3-Spider



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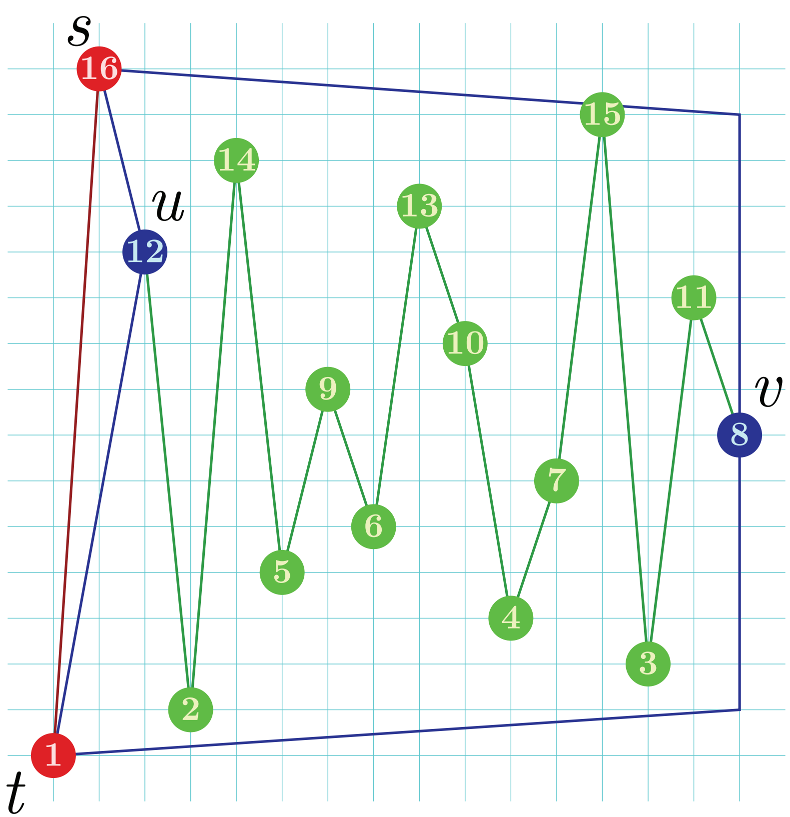

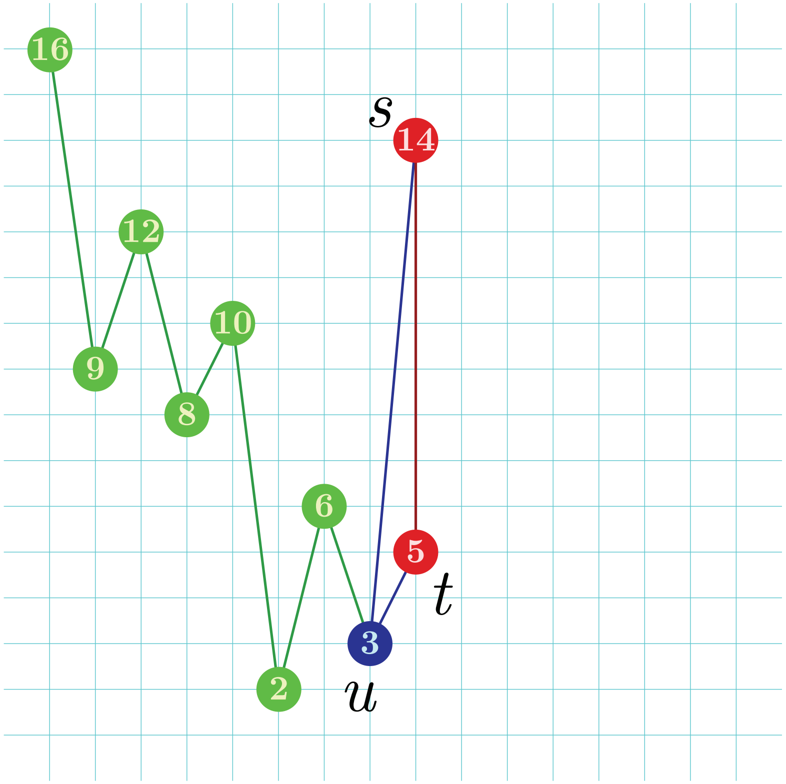

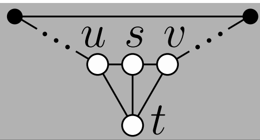

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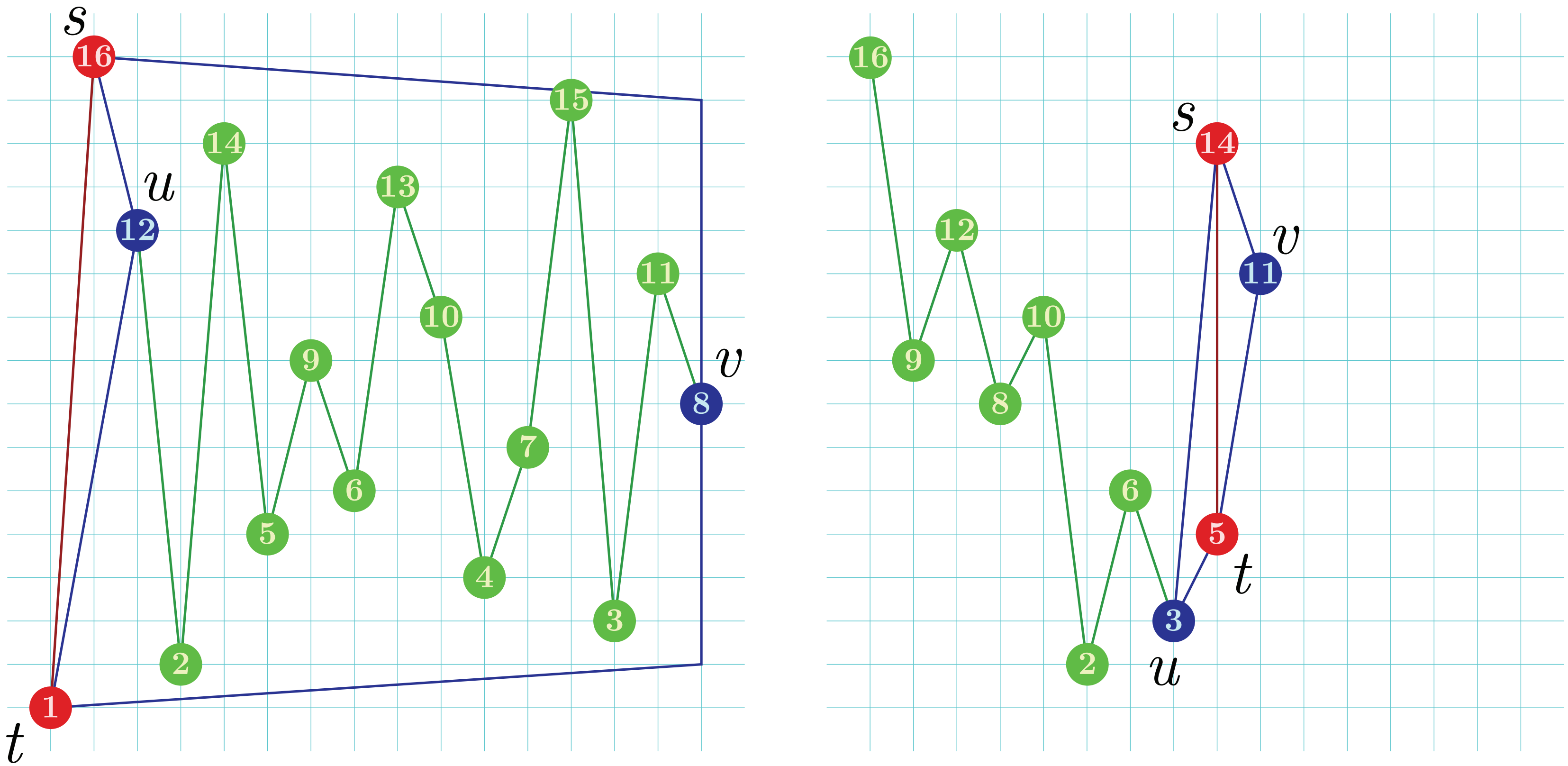

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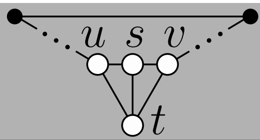

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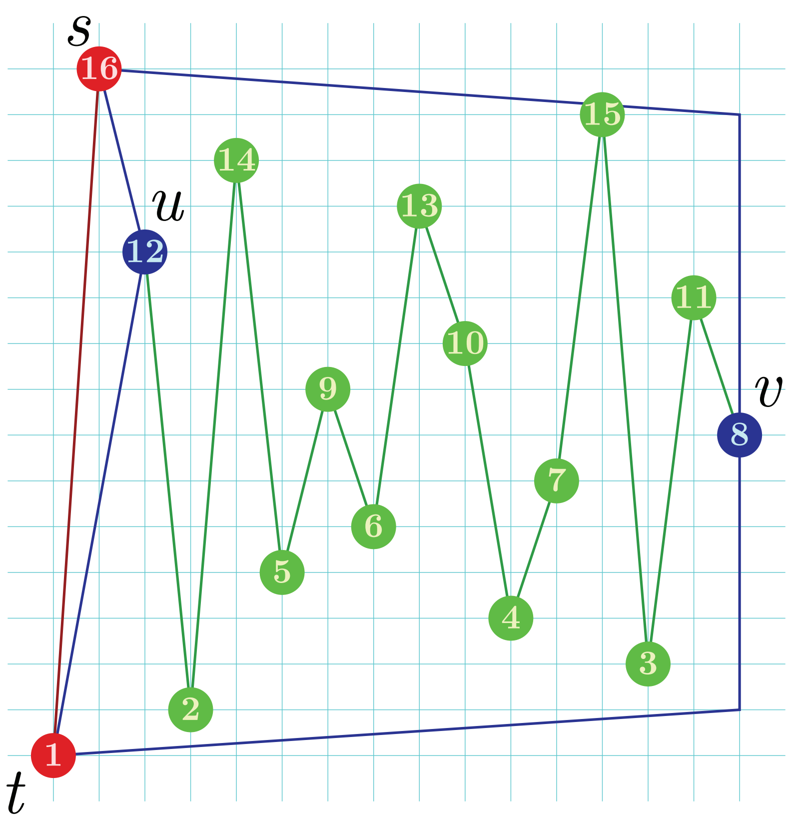

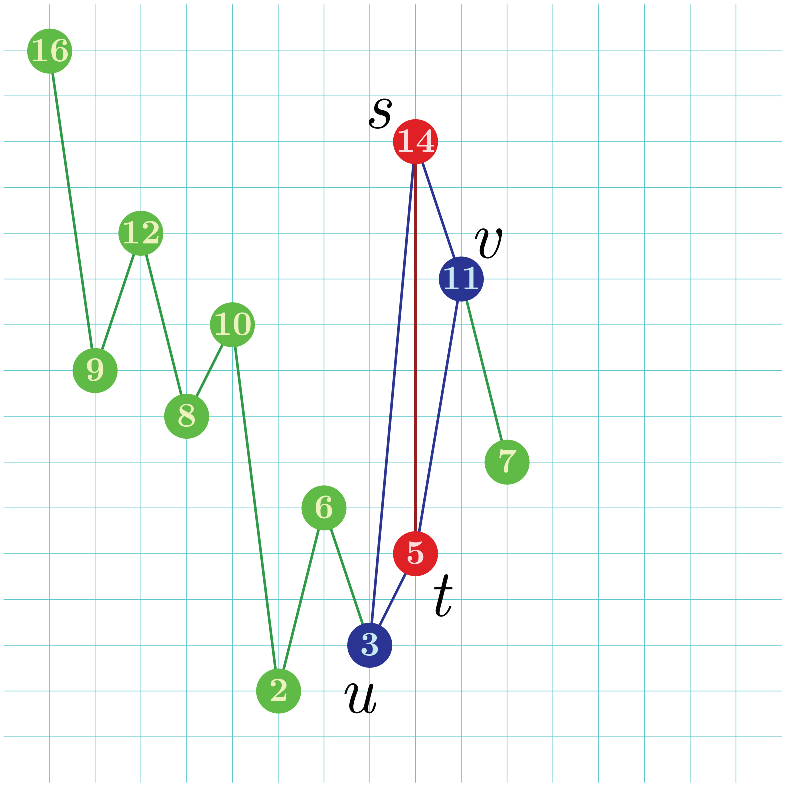

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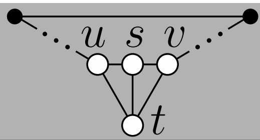

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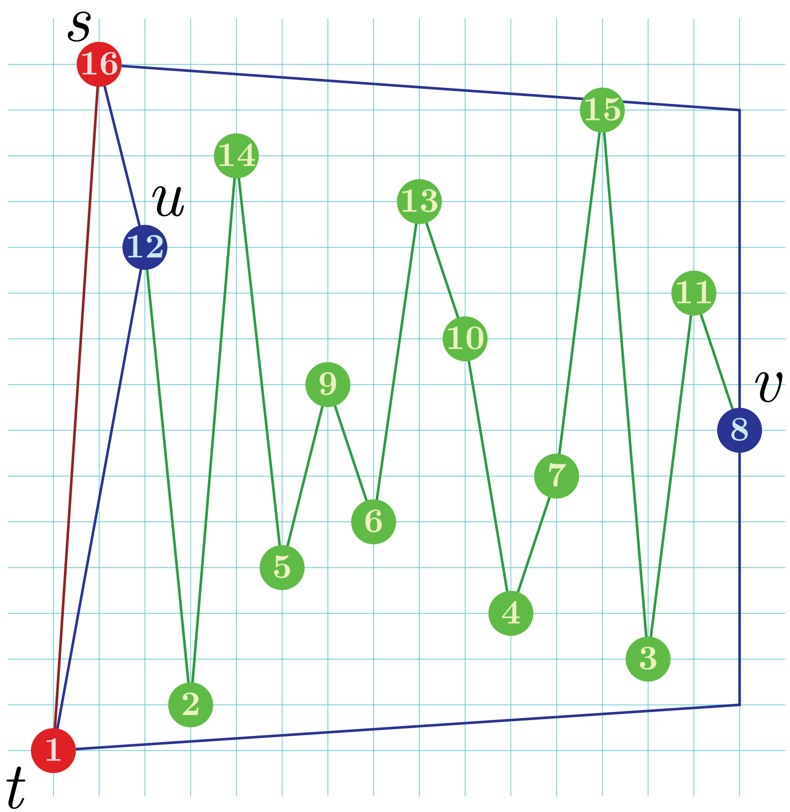

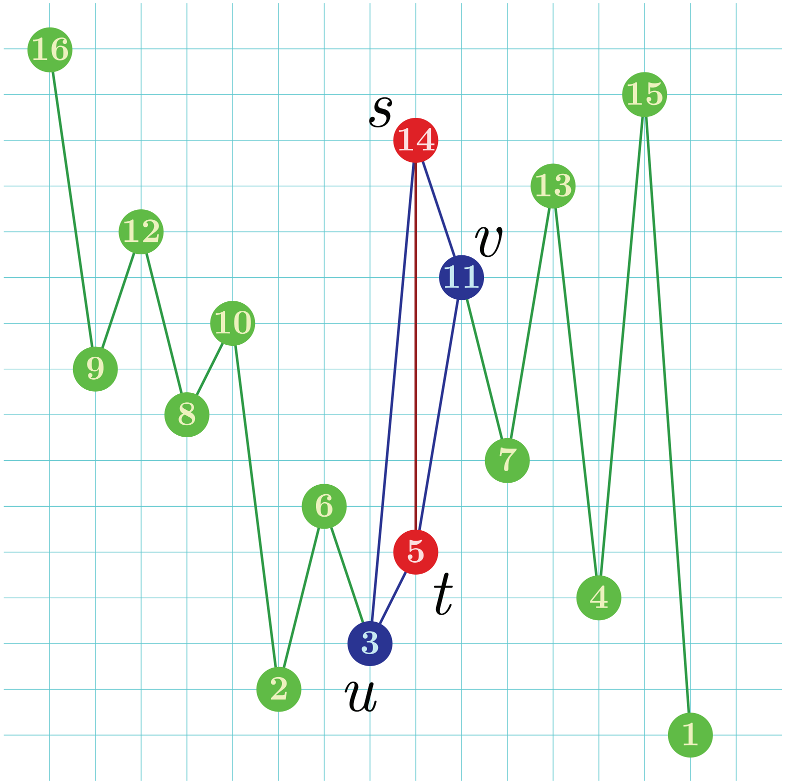

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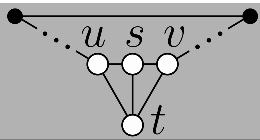

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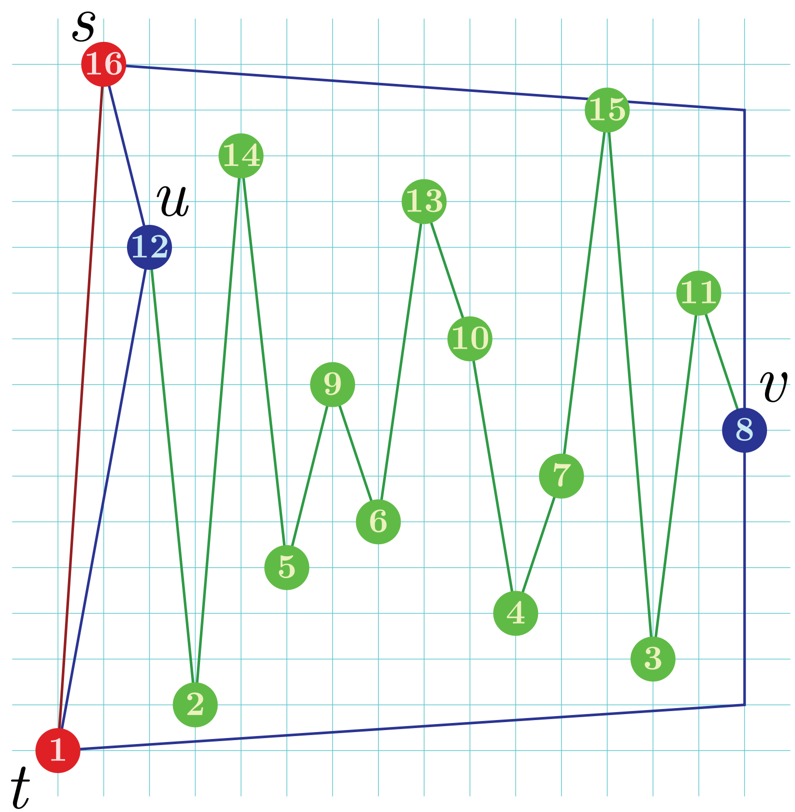
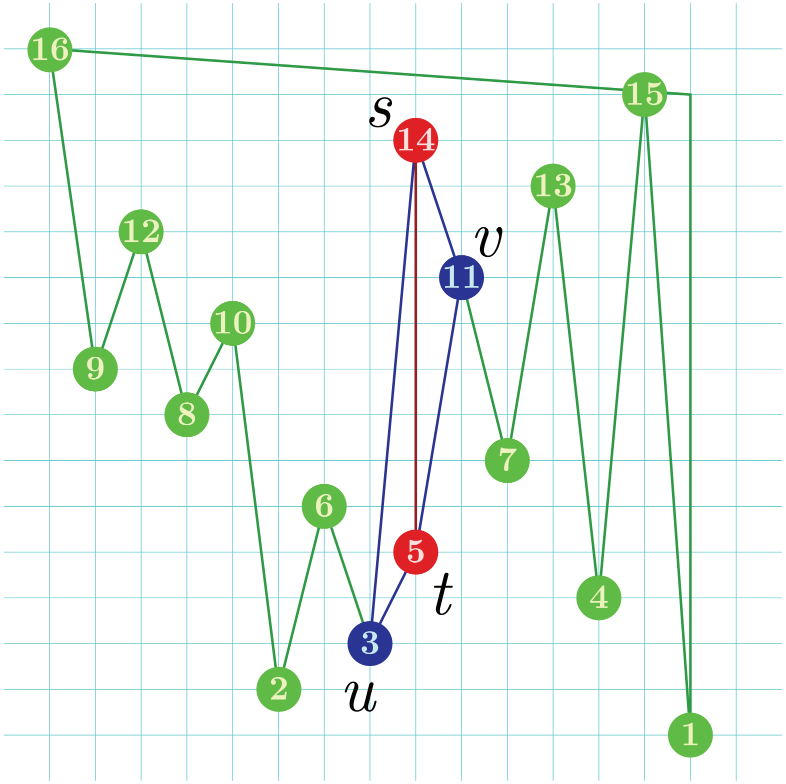
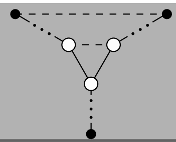


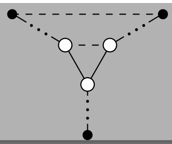


1-Connected Extended 3-Spider



- Drawing a 1-connected extended degree-3 spider:

1-Connected Extended 3-Spider



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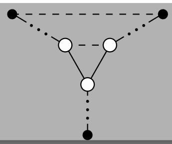

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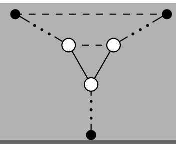
- Proof idea:
  - Modify degree-3 spider algorithm, same invariants

1-Connected Extended 3-Spider



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  - Break large cycle so one endpoint is extreme

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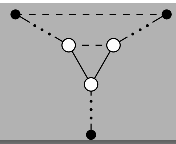

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- Proof idea:
  - Modify degree-3 spider algorithm, same invariants
  - Break large cycle so one endpoint is extreme
  - Getting degree-3 spider started is more involved

1-Connected Extended 3-Spider

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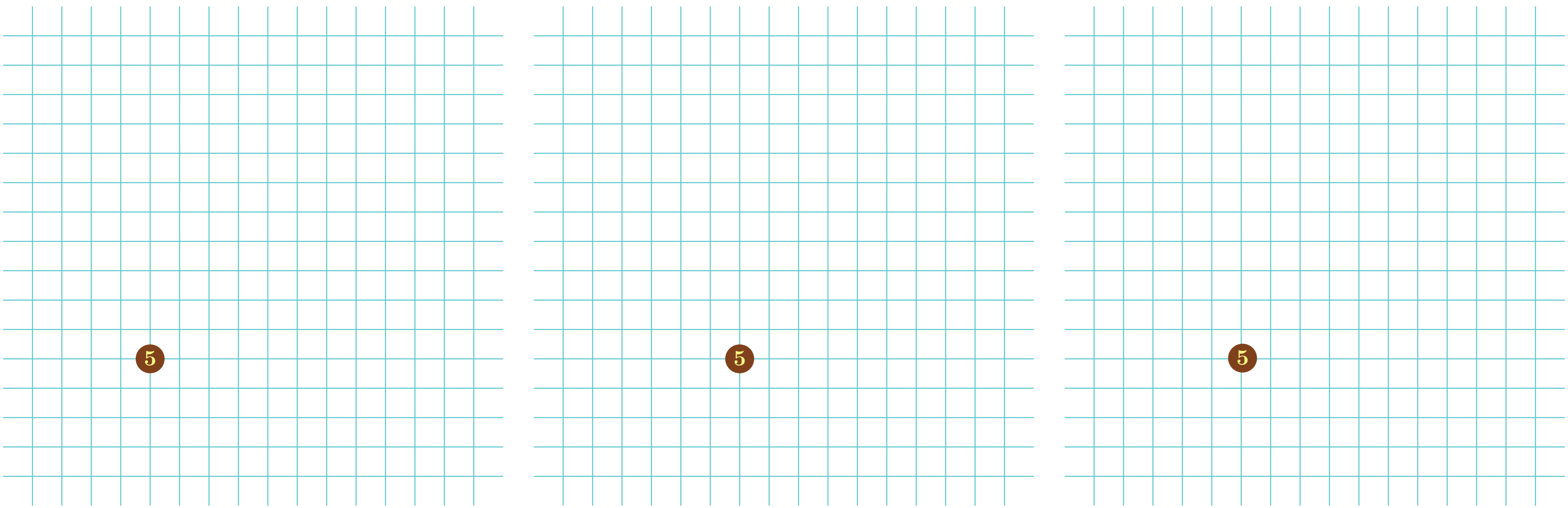
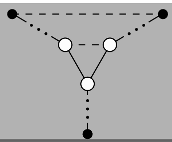


  - Two cases of starting spider for extra edge that forms a K_3

1-Connected Extended 3-Spider



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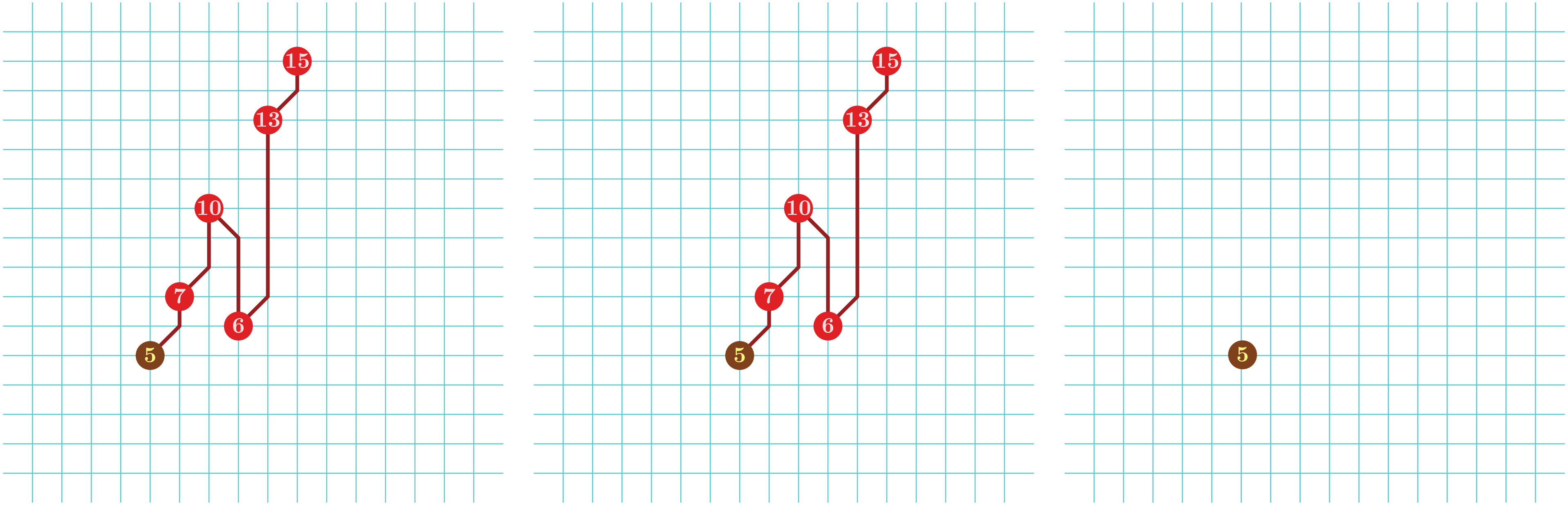
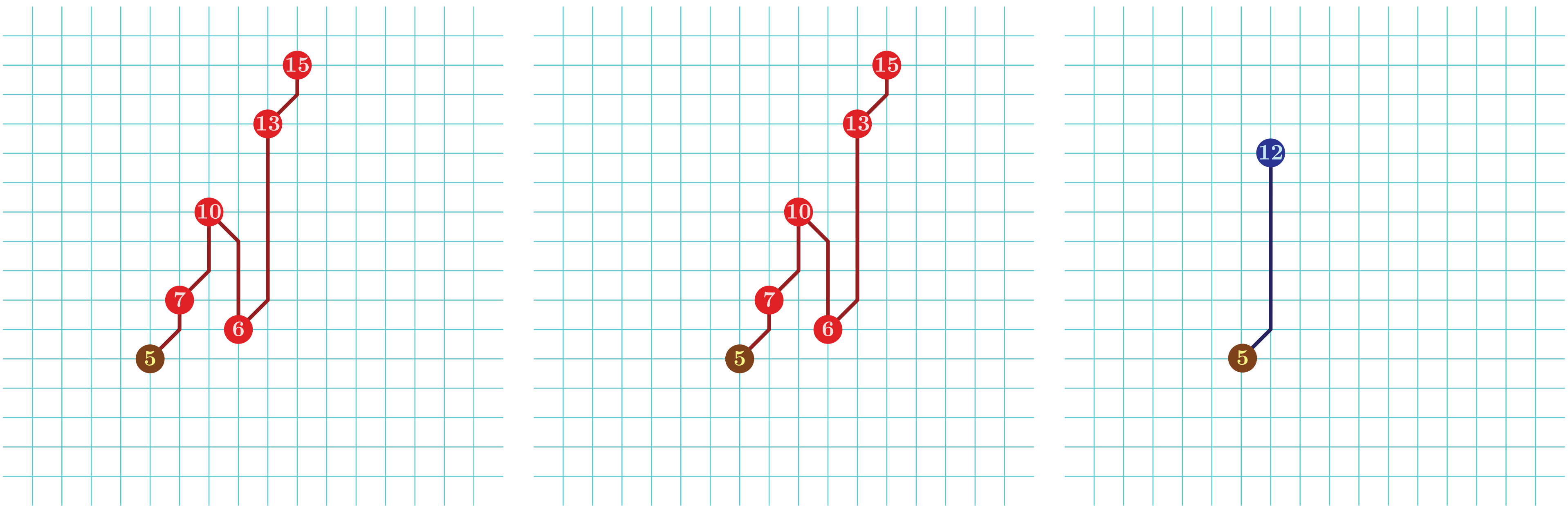


  - In the first case red path is first drawn to its maximum

1-Connected Extended 3-Spider

- Drawing a 1-connected extended degree-3 spider:

  Lemma 10 *A planar drawing of an n-vertex 1-connected extended degree-3 spider $G(V, E)$ can be drawn in $O(n)$ time on an $n \times n$ grid for any vertex labeling $\phi : V \xrightarrow[\text{onto}]{1:1} \{1, 2, \ldots, n\}$ with one bend per edge.*



  - In the second case blue path is drawn first

1-Connected Extended 3-Spider

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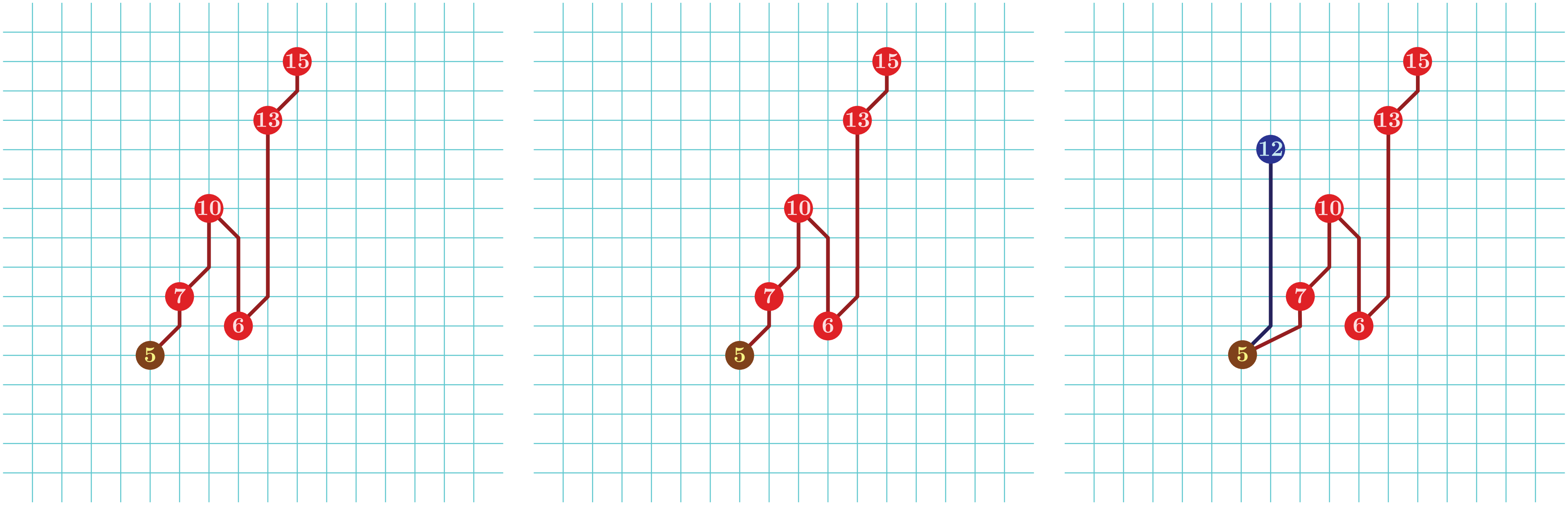
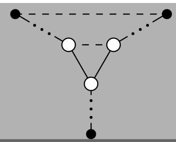


  - Then the red path is drawn next

1-Connected Extended 3-Spider



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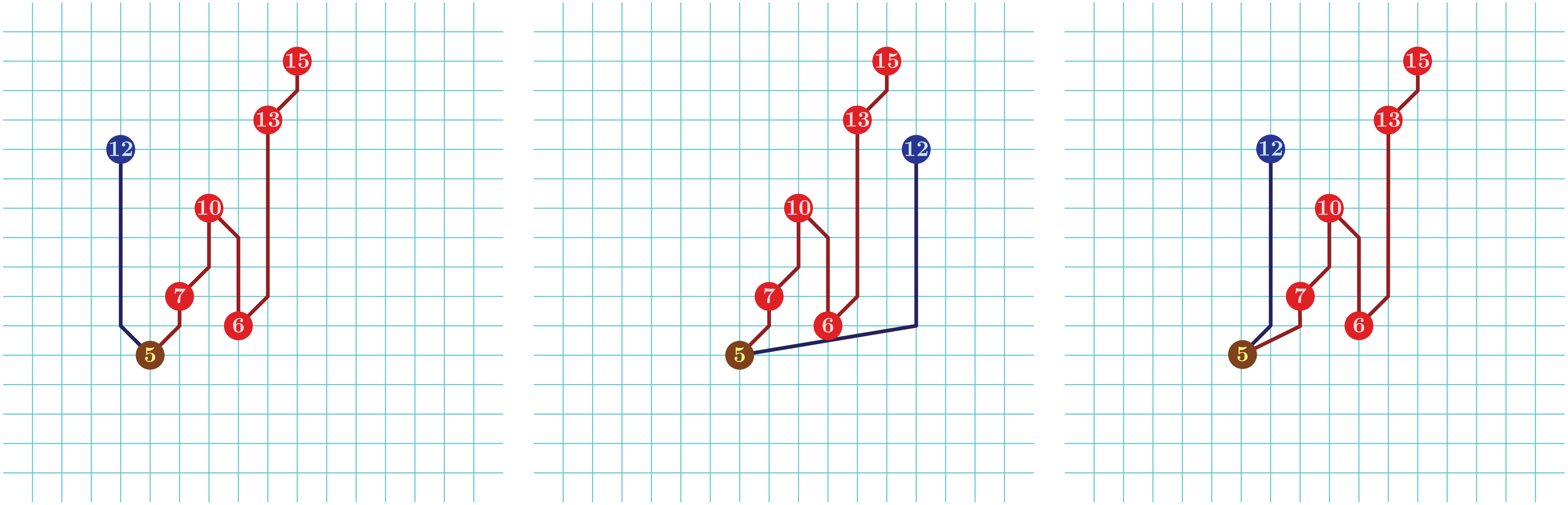
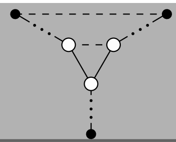


  - Whereas in the first case the blue path comes second

1-Connected Extended 3-Spider



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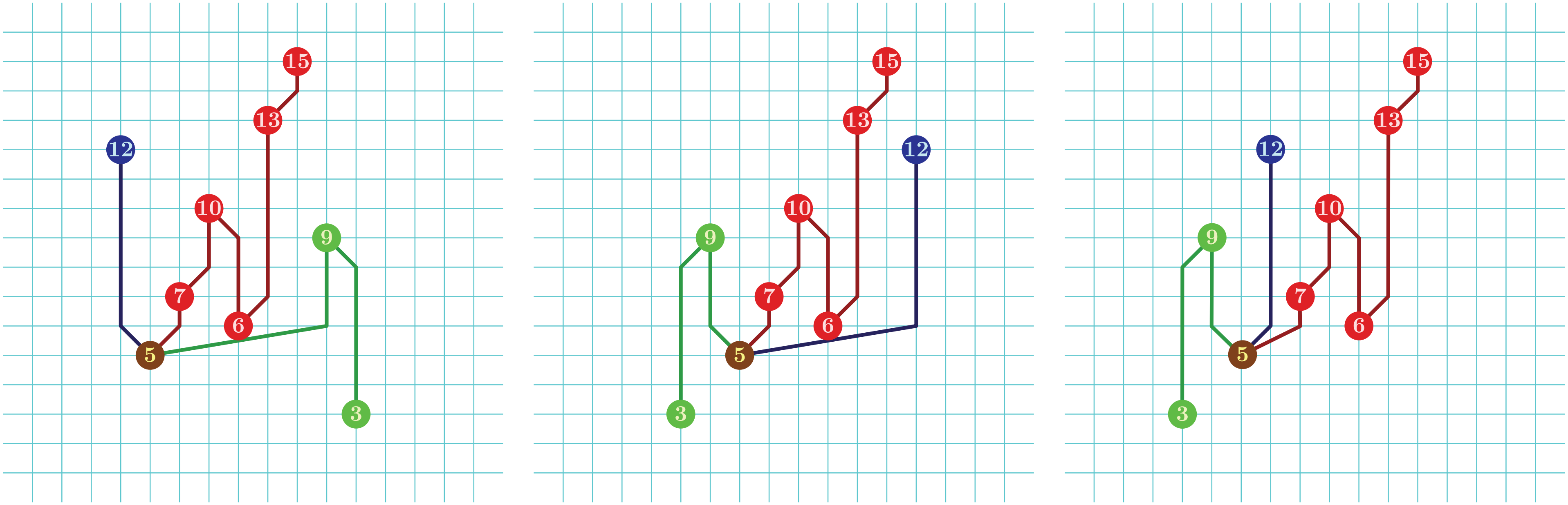
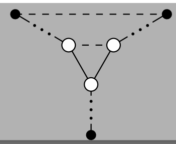


  - Then the green paths are drawn to their minimum

1-Connected Extended 3-Spider



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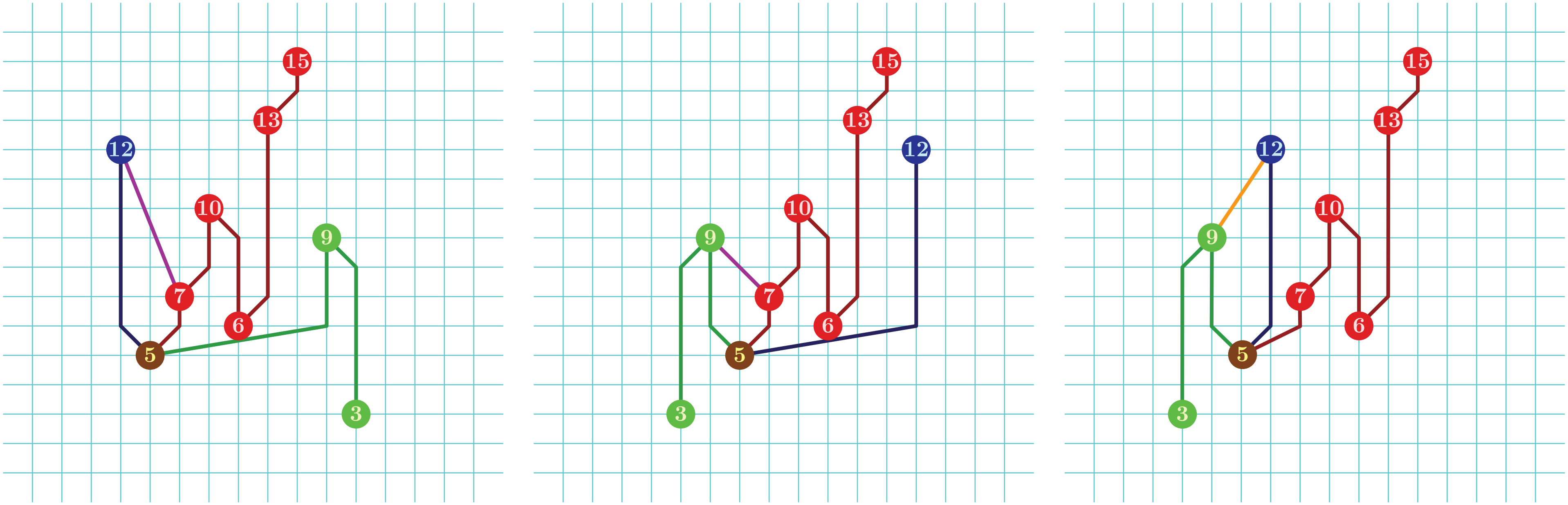
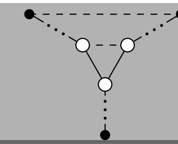


  - Can then accommodate the extra edge

1-Connected Extended 3-Spider



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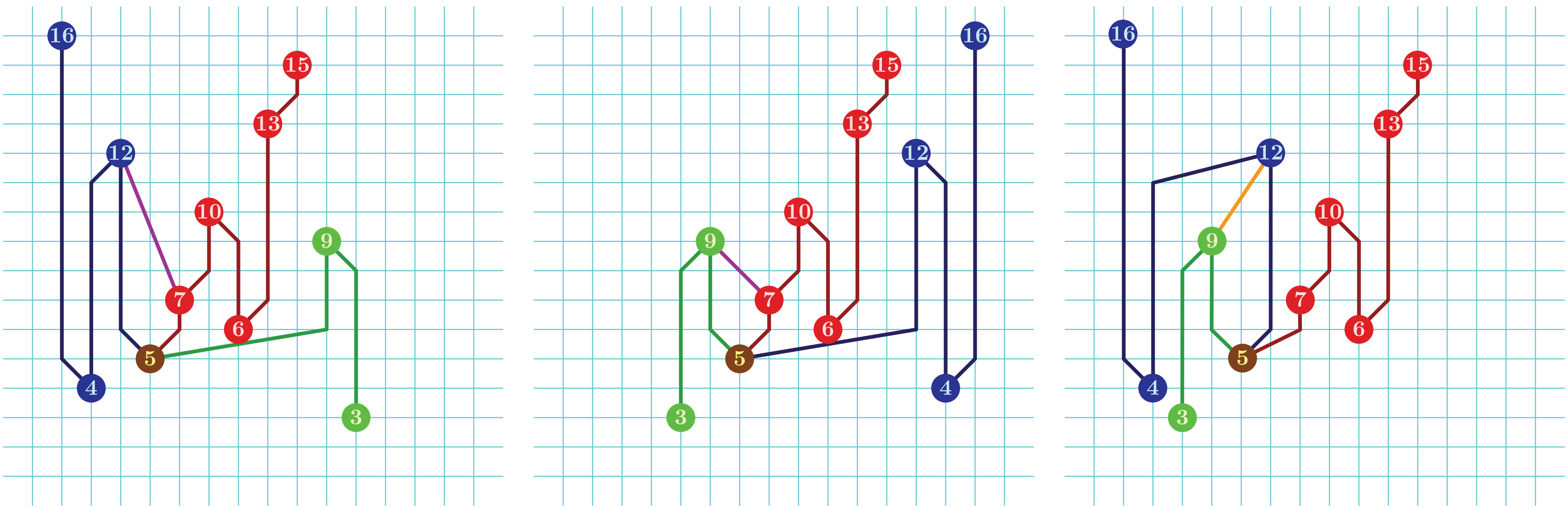

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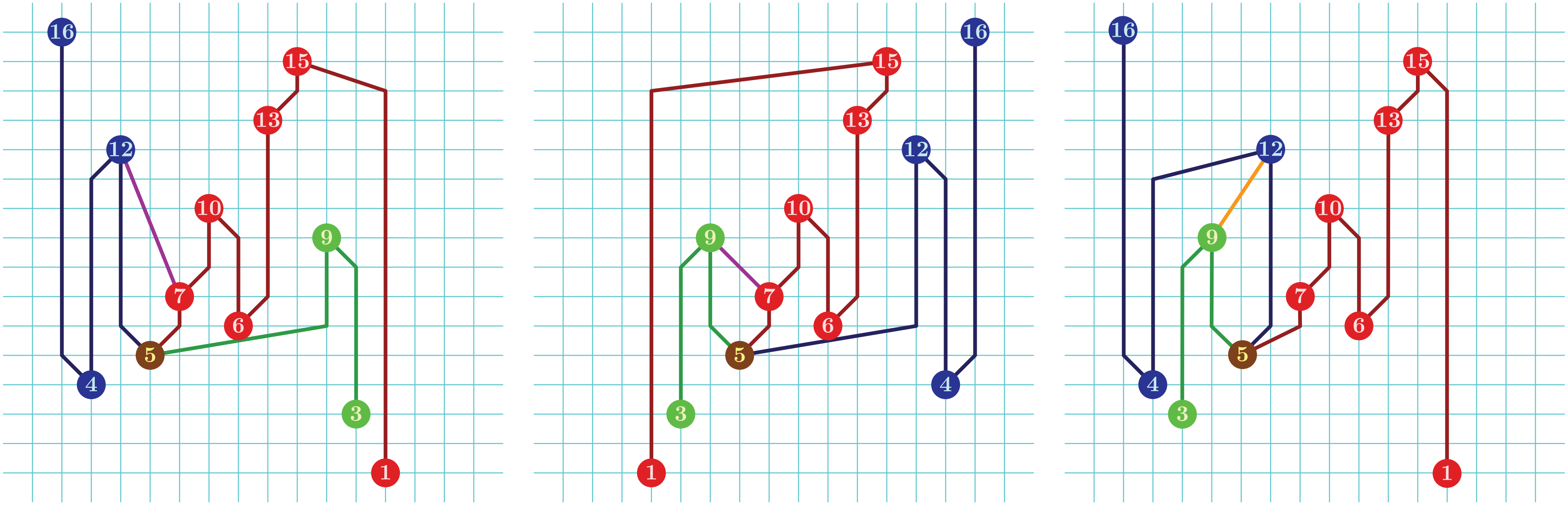

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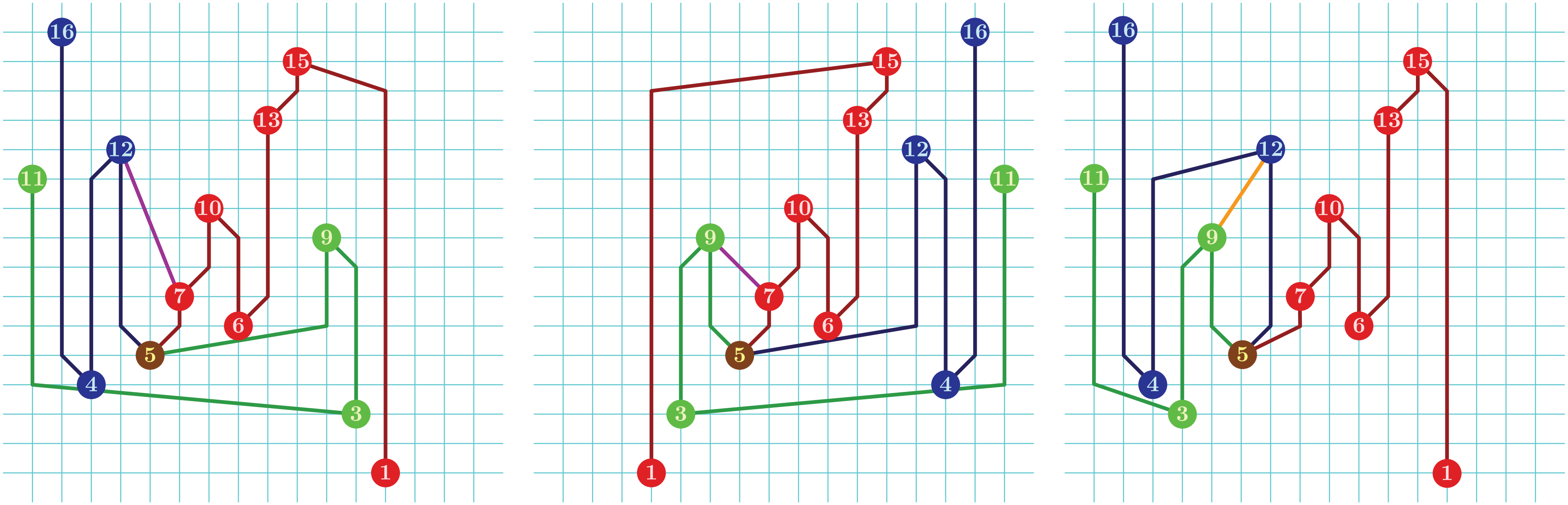

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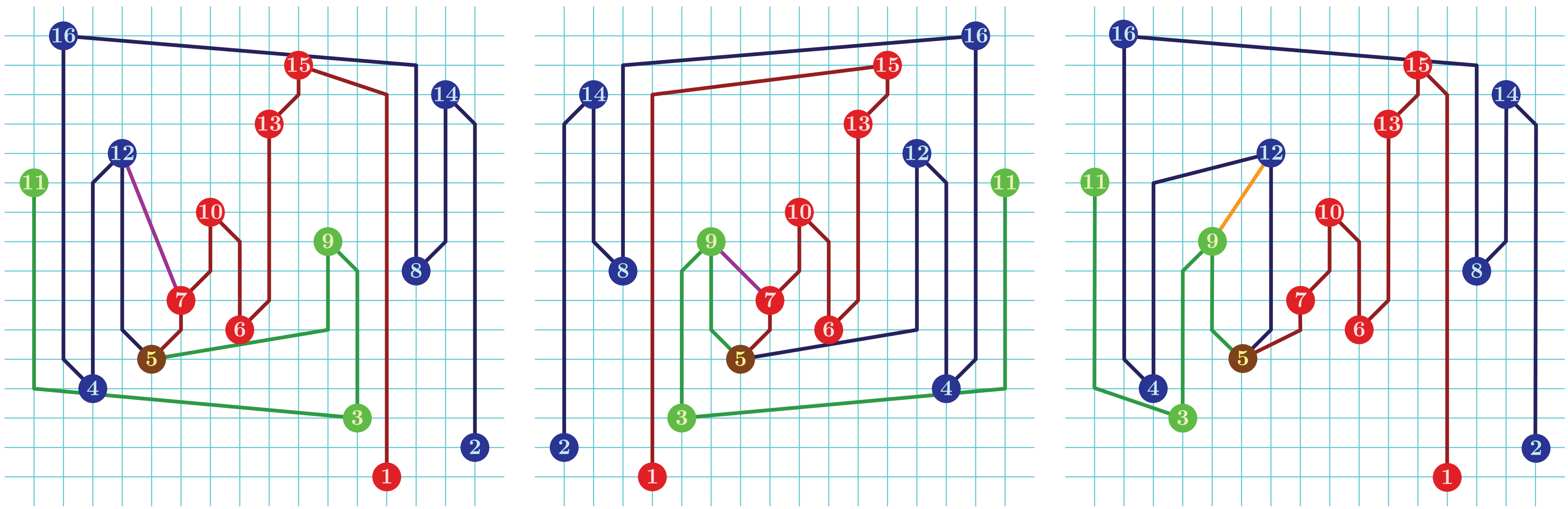
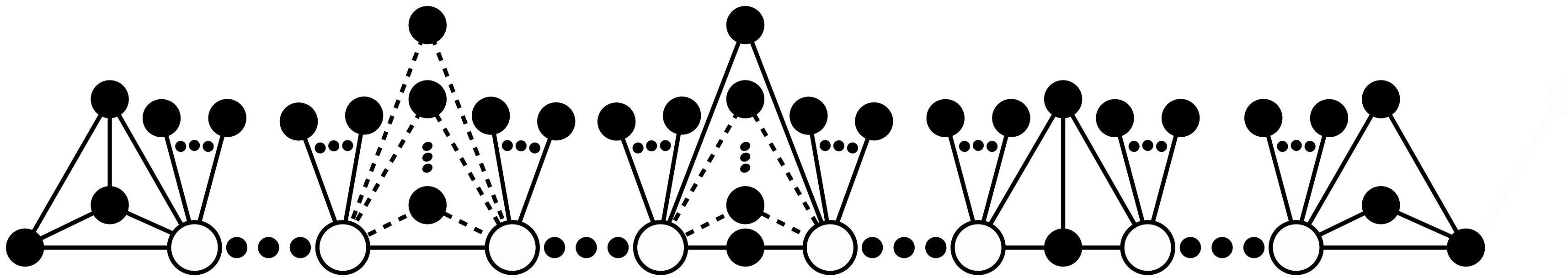


  - ► Can then accommodate the extra edge

Characterization – ULP Graphs

- Proof by induction gives the next theorem:

  Theorem 11 *Every graph either contains one of the seven forbidden graphs in which case it is not ULP, or it is a generalized caterpillar, a radius-2 star, or a extended degree-3 spider in which case it is ULP.*



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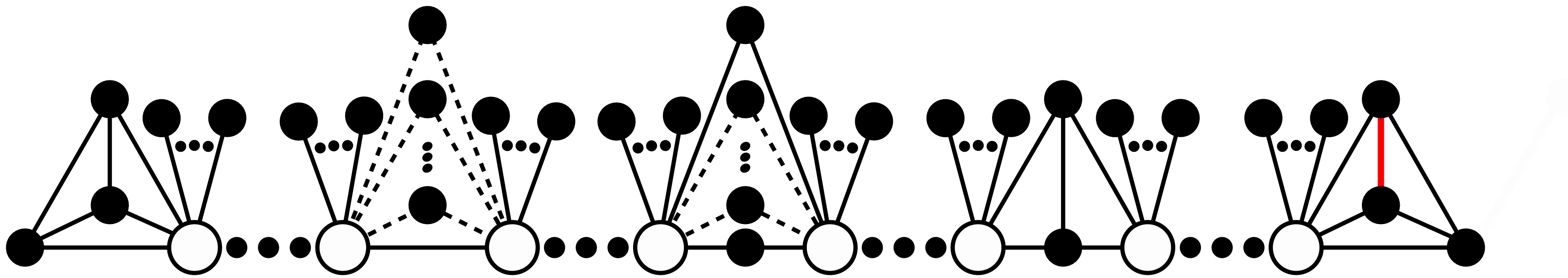
- Proof idea:
  - Assume graph with n edges is ULP



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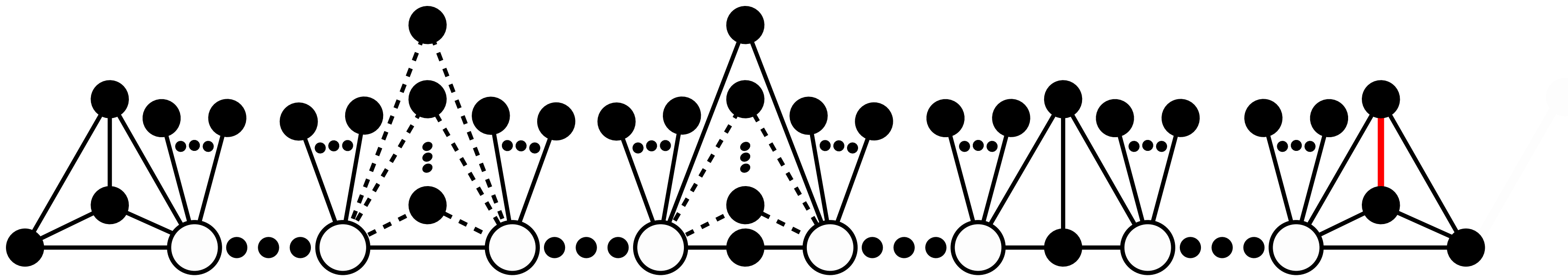
- Proof idea:
  - Consider all possible ways of adding an extra edge

Characterization – ULP Graphs



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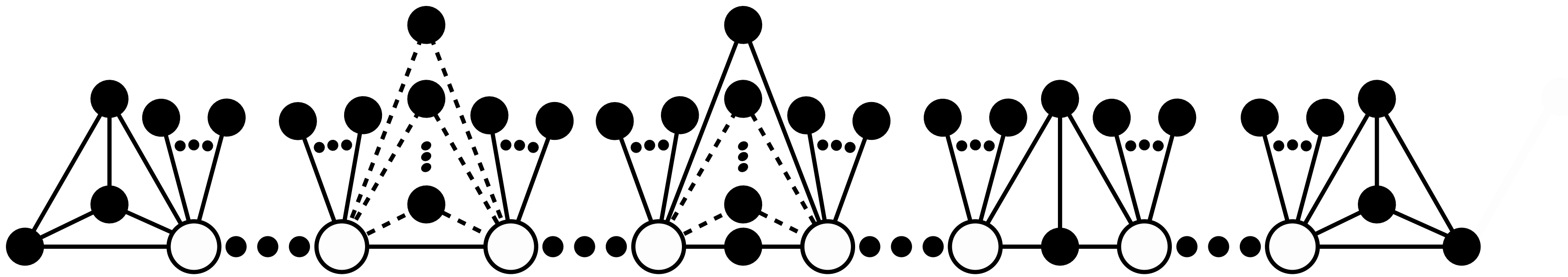
- Proof idea:
  - Either two possibilities



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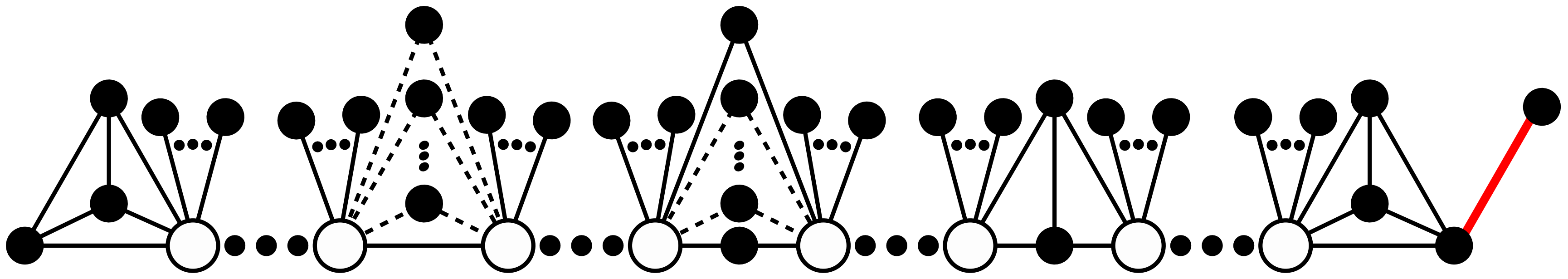
- Proof idea:
  - Either two possibilities
    - Graph remains ULP

Characterization – ULP Graphs



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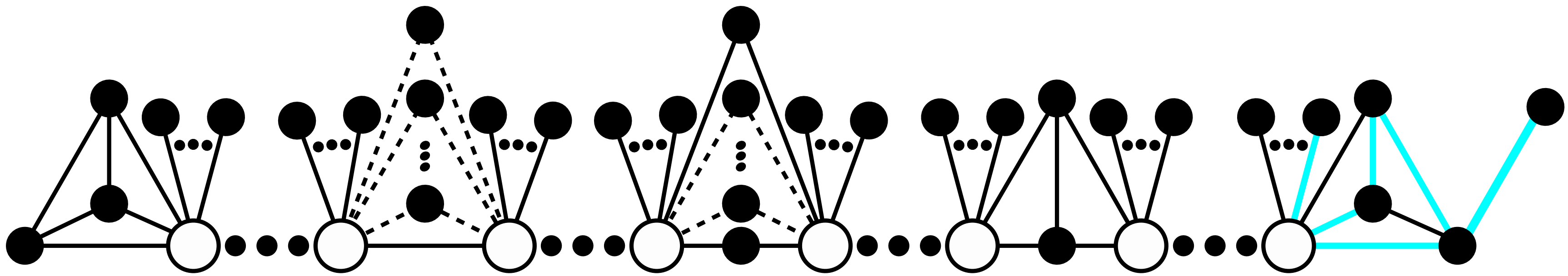
- Proof idea:
  - Either two possibilities
    - OR



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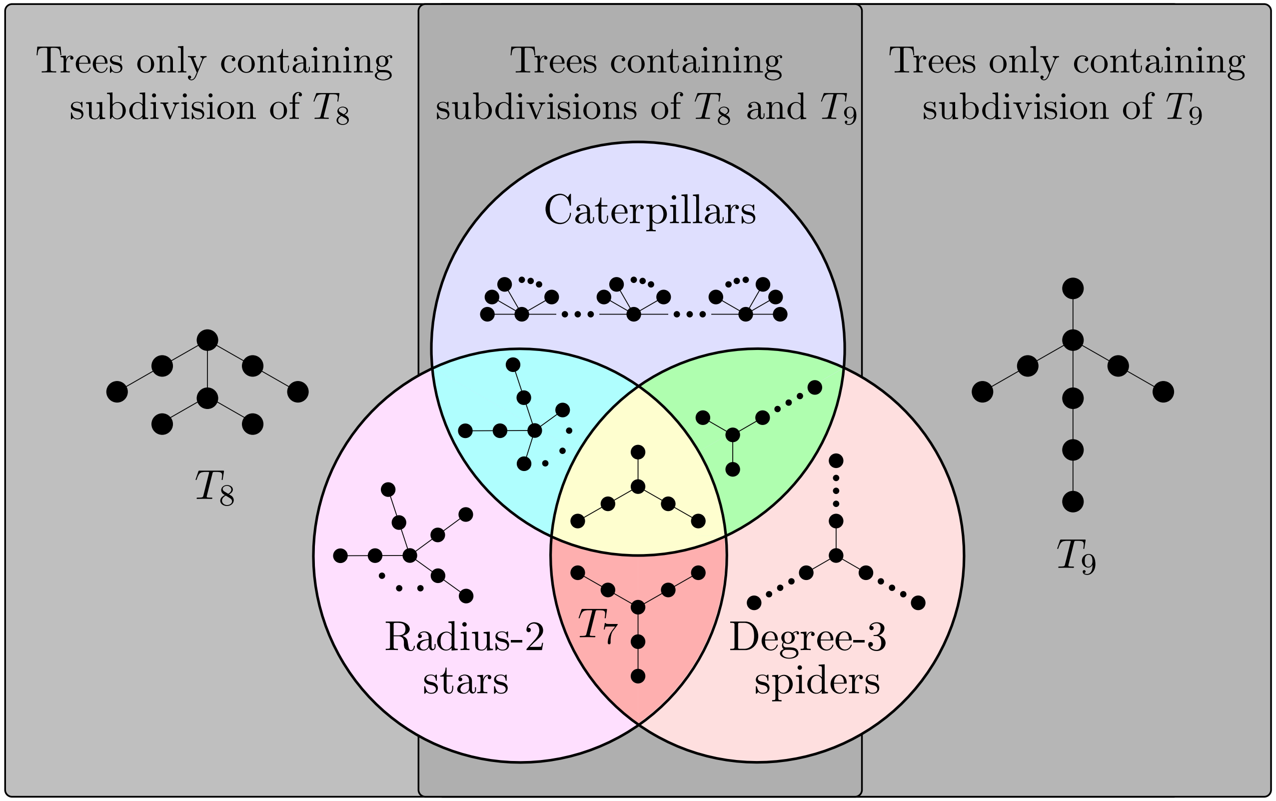
- Proof idea:
  - Either two possibilities
    - Graph now contains one of the forbidden graphs – G_6 in this case

Summary

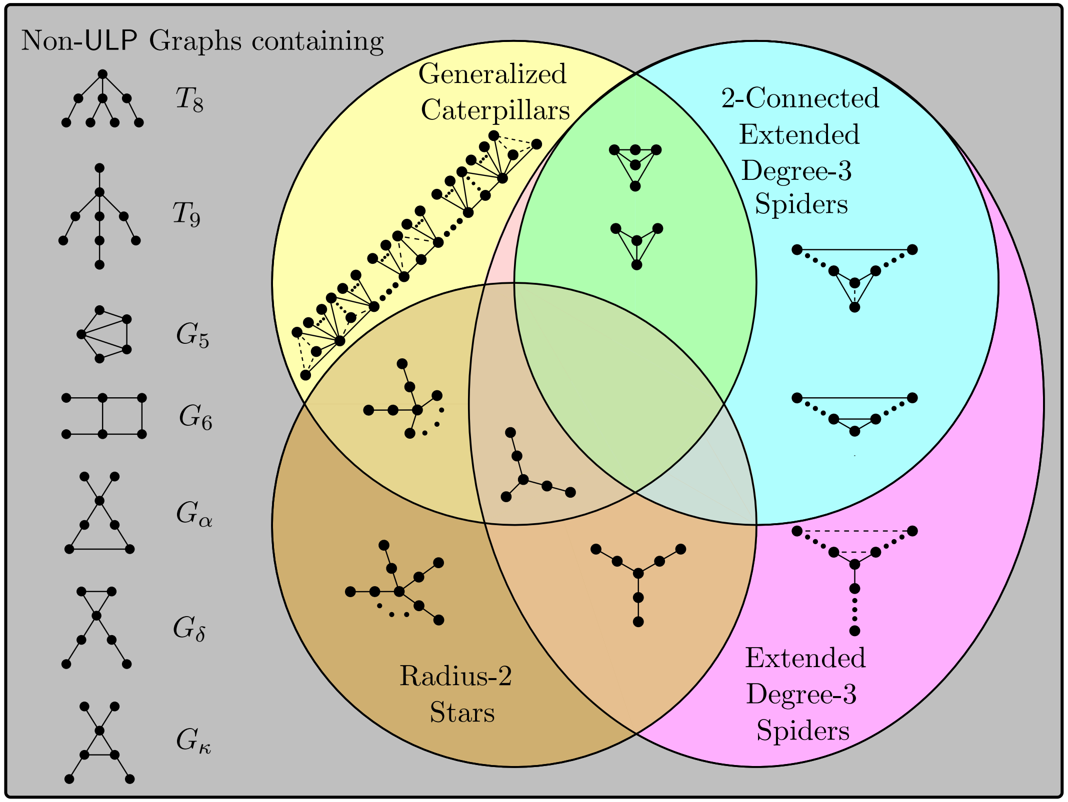
- Background
- Unlabeled Level Planar Trees



  - Previous results for trees

Summary

- Background
- Unlabeled Level Planar Trees
- Unlabeled Level Planar Graphs



  - New results for all graphs

Future Work

- Provide recognition algorithm for all ULP graphs

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- Provide certificate of unlabeled level non-planarity

Future Work

- Provide recognition algorithm for all ULP graphs
- Provide certificate of unlabeled level non-planarity
  - I.e., find a copy of a forbidden ULP graph

Thank You!