Characterization of Unlabeled Level Planar Trees

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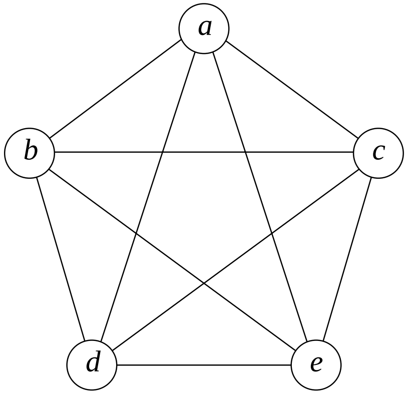
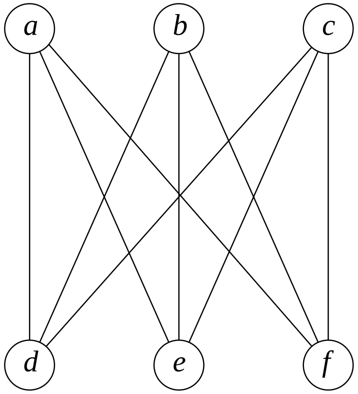


K_5



$K_{3,3}$

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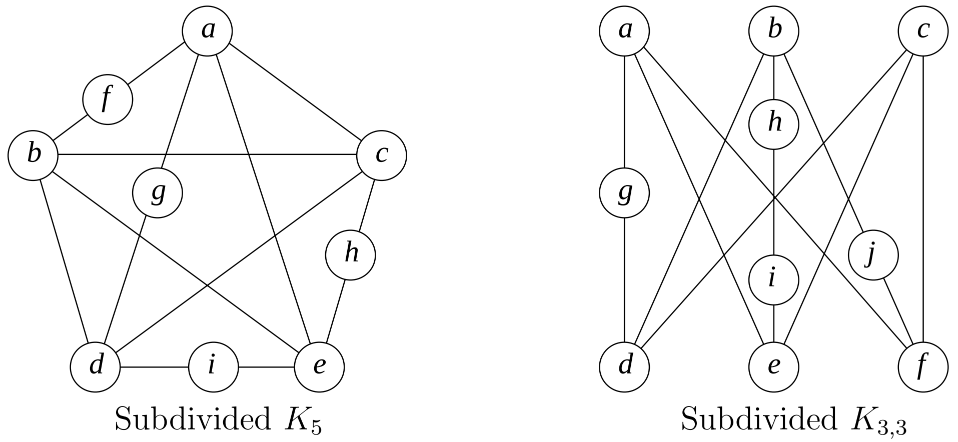

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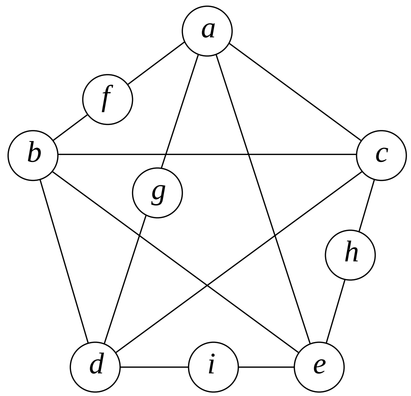
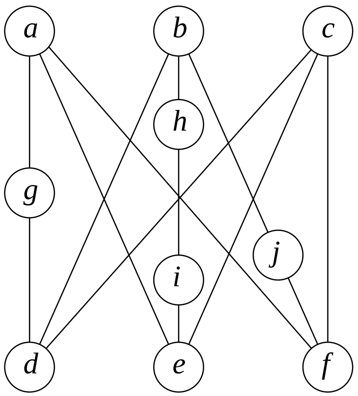


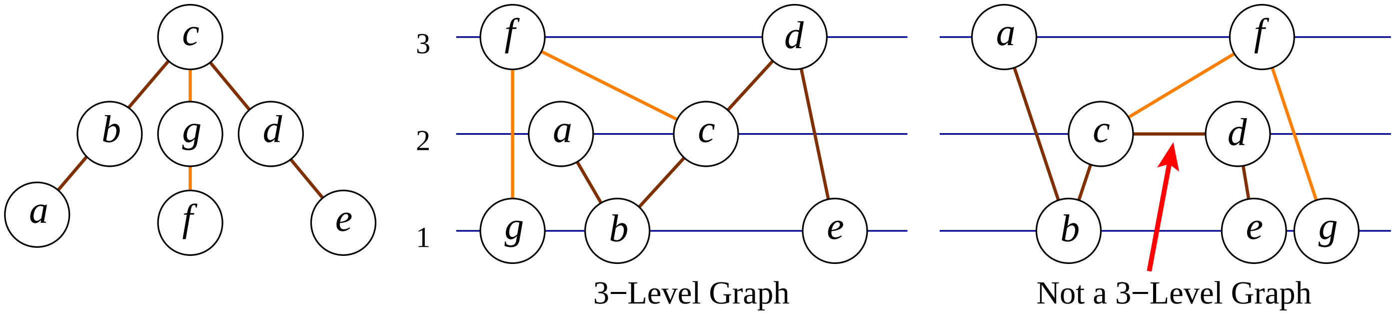
Subdivided K_5

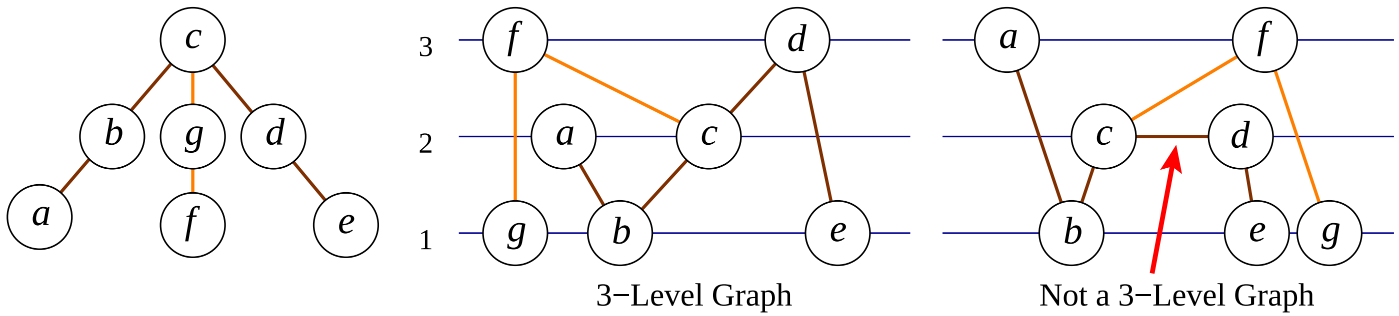


Subdivided $K_{3,3}$

    - I.e., G does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$
- Have developed similar forbidden subdivision characterization for ULP trees

Level Planarity – Definitions





- A k-*level graph* $G(V, E, \phi)$
  - Has n vertices where $n \geq k$
  - Edges are drawn with straight-line segments
  - Has a *level assignment* $\phi : V \longrightarrow [1..k]$
    - Assigns each vertex to one of k equidistant horizontal levels
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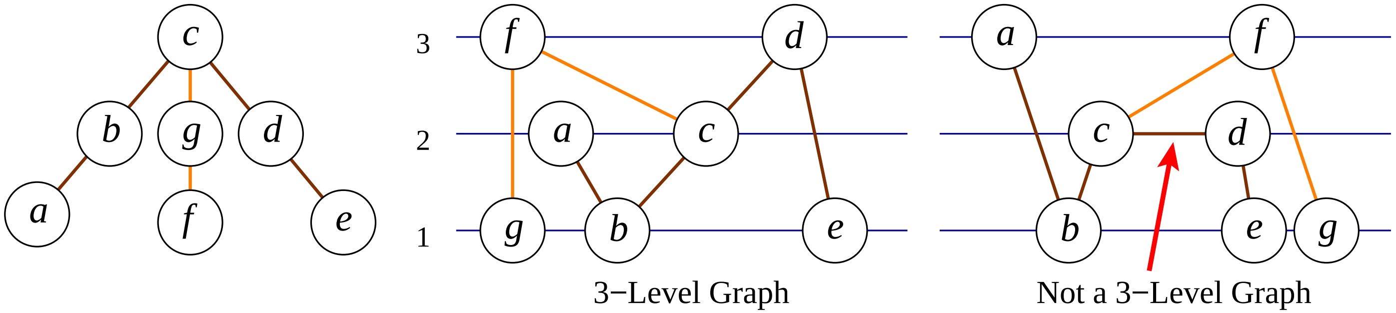

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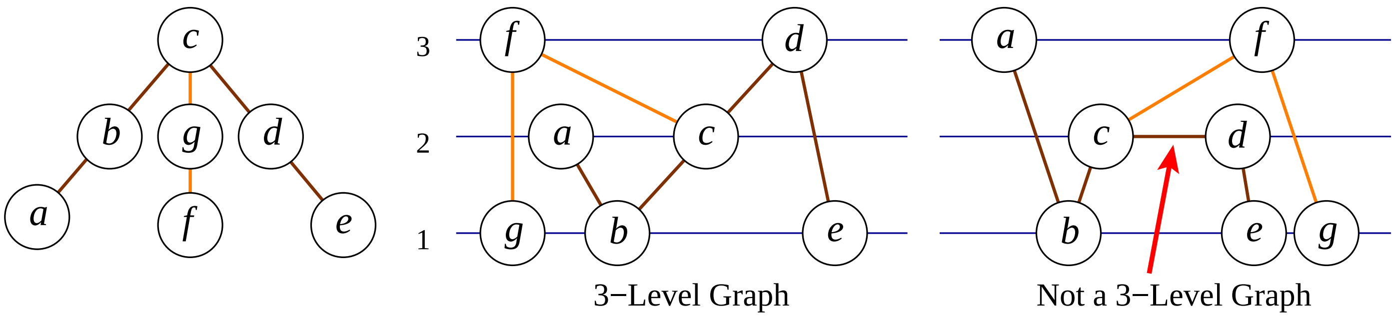

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  - Such a plane drawing forms a *realization* of G

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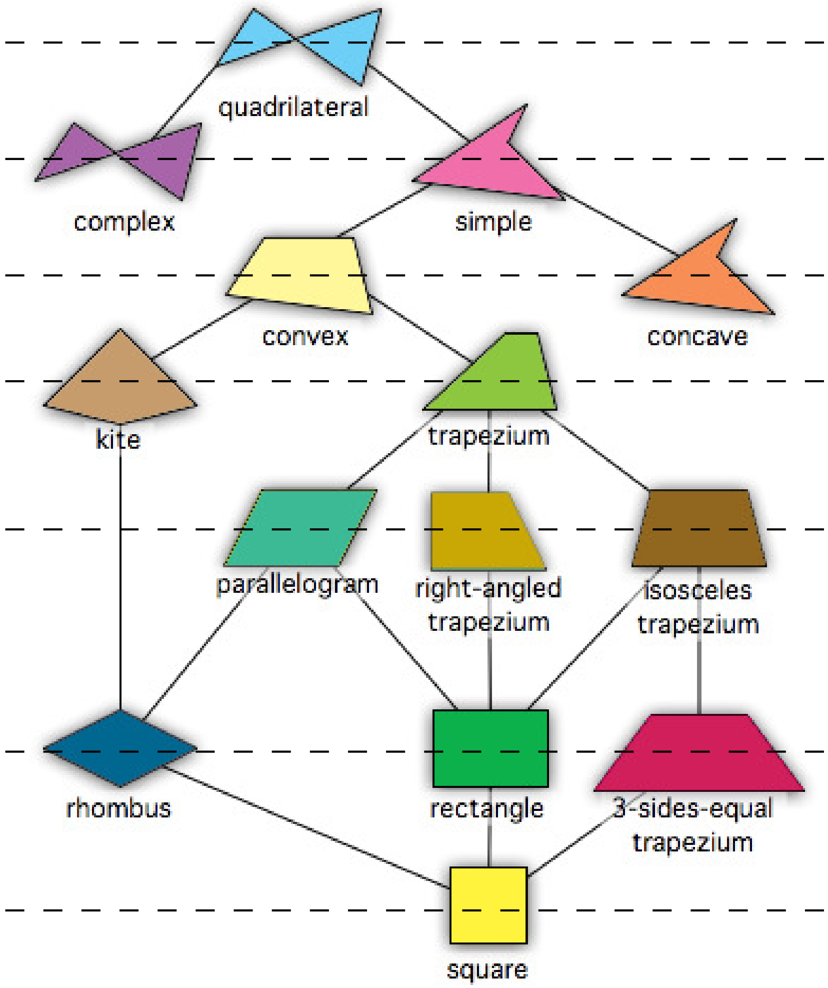

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  - Finding a k-level assignment for which a graph is level planar is NP-hard

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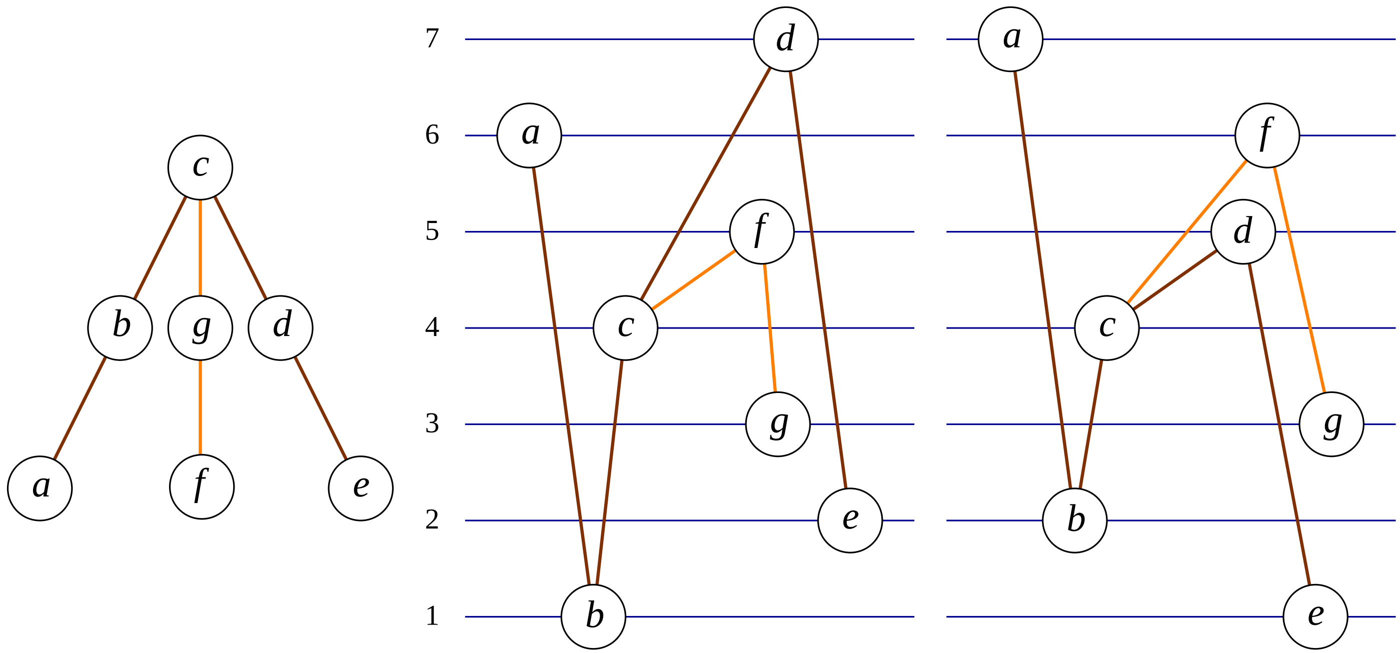

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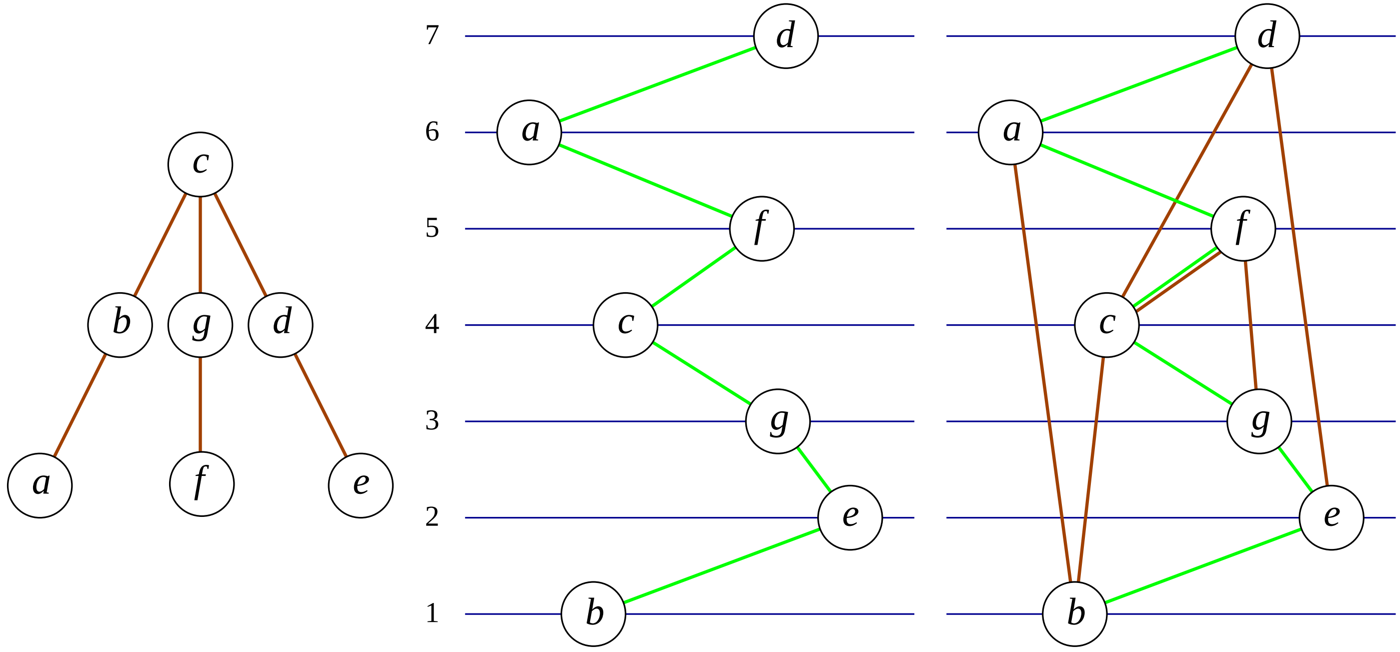
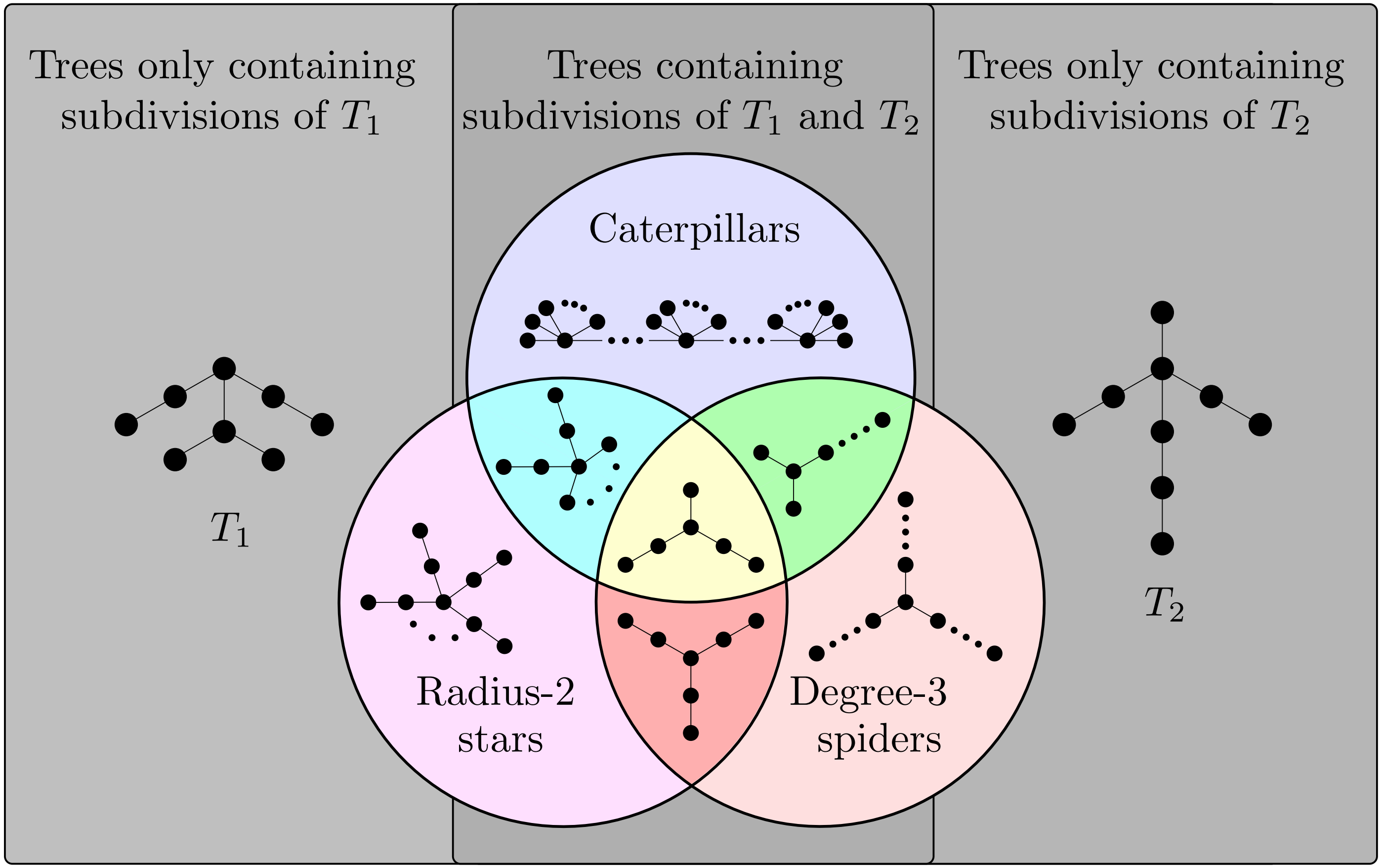
  - Embedding of multiple planar graphs onto the same vertex set V
    - Has to work for *any* vertex mapping between graphs
    - Desire straight-line edges
    - No crossings allowed within each planar layer
  - Can simultaneously embed a path P with any ULP graph G



Unlabeled Level Planarity – Trees



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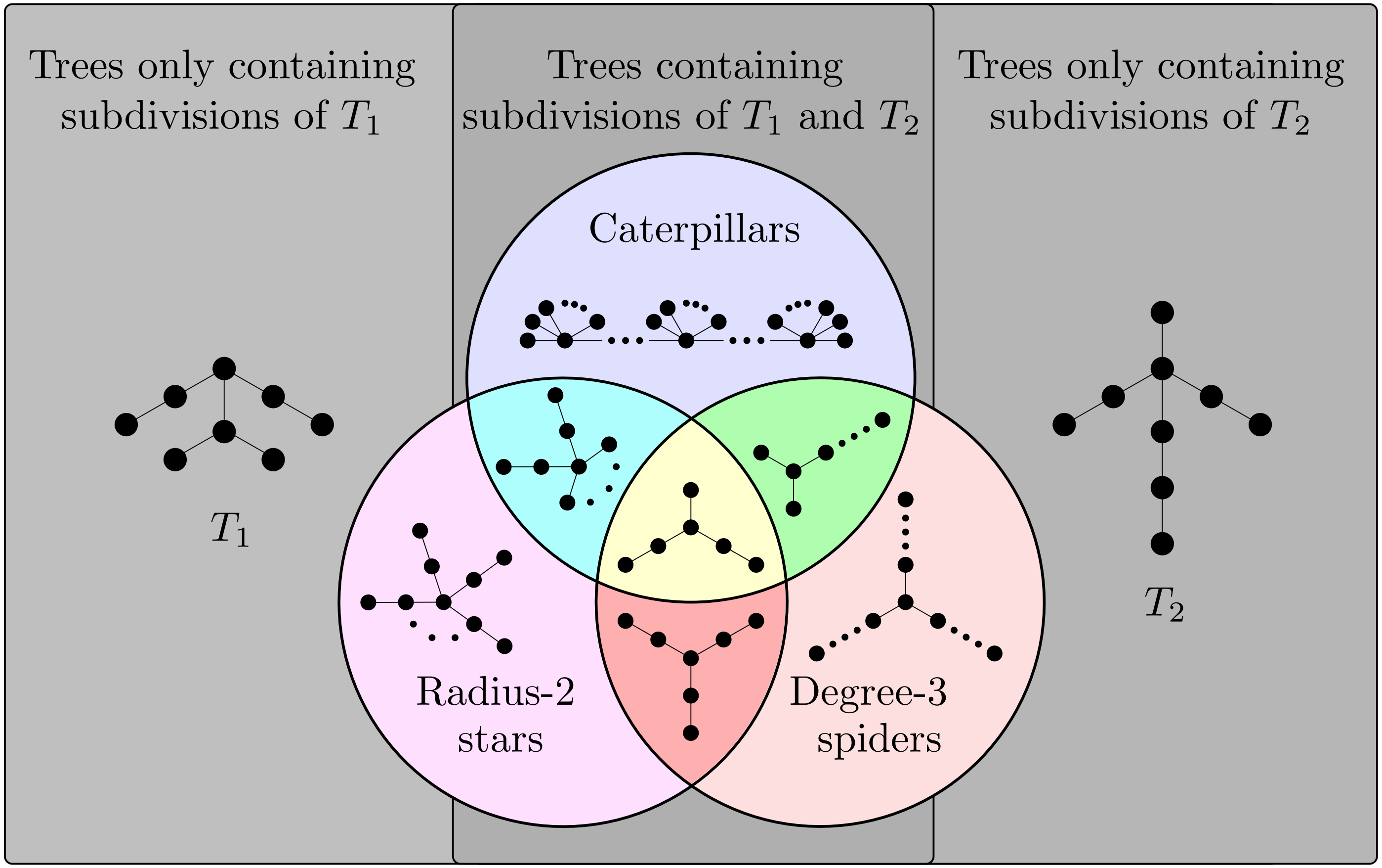

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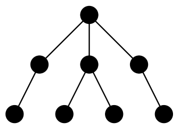

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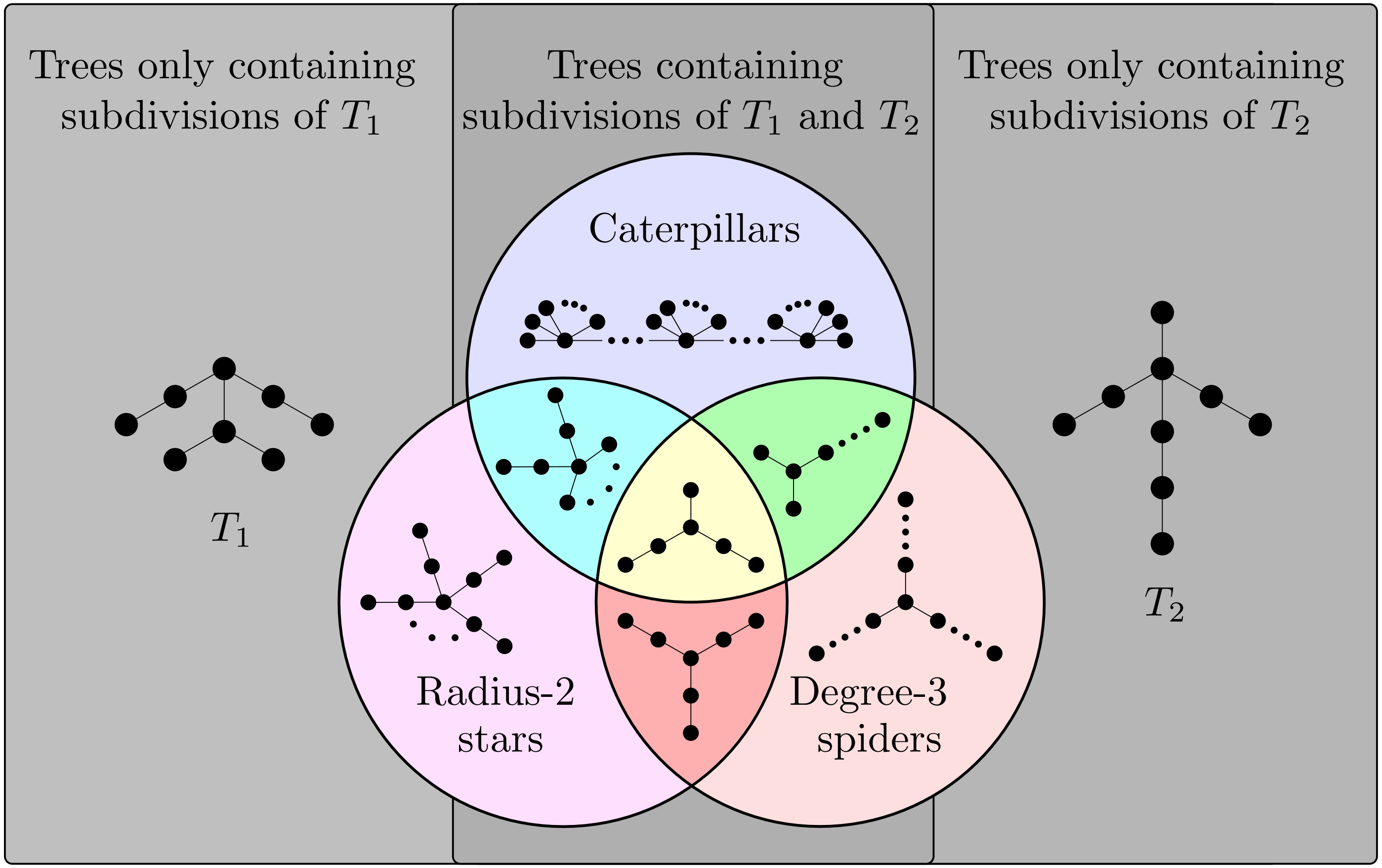

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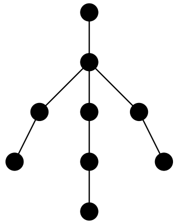

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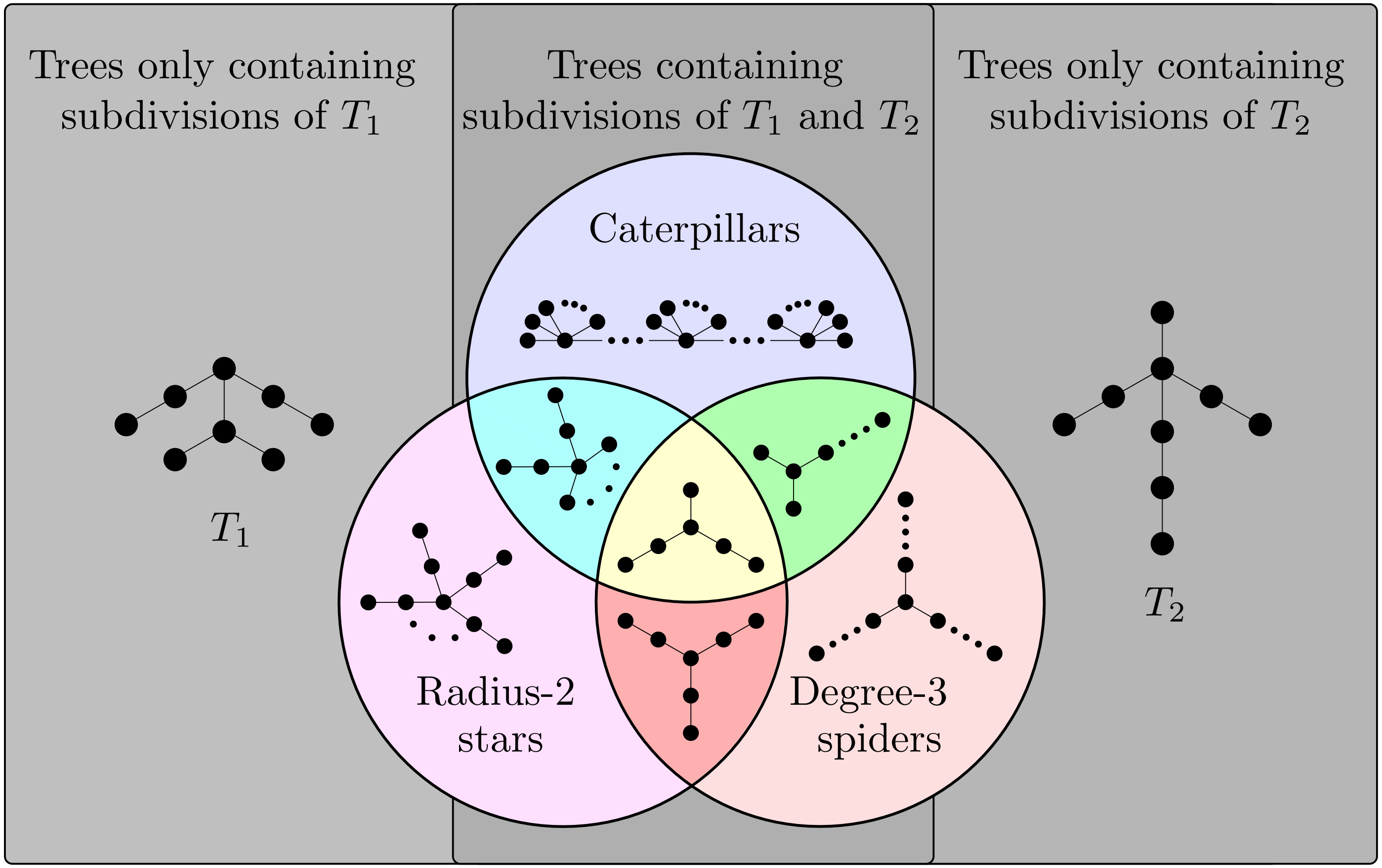

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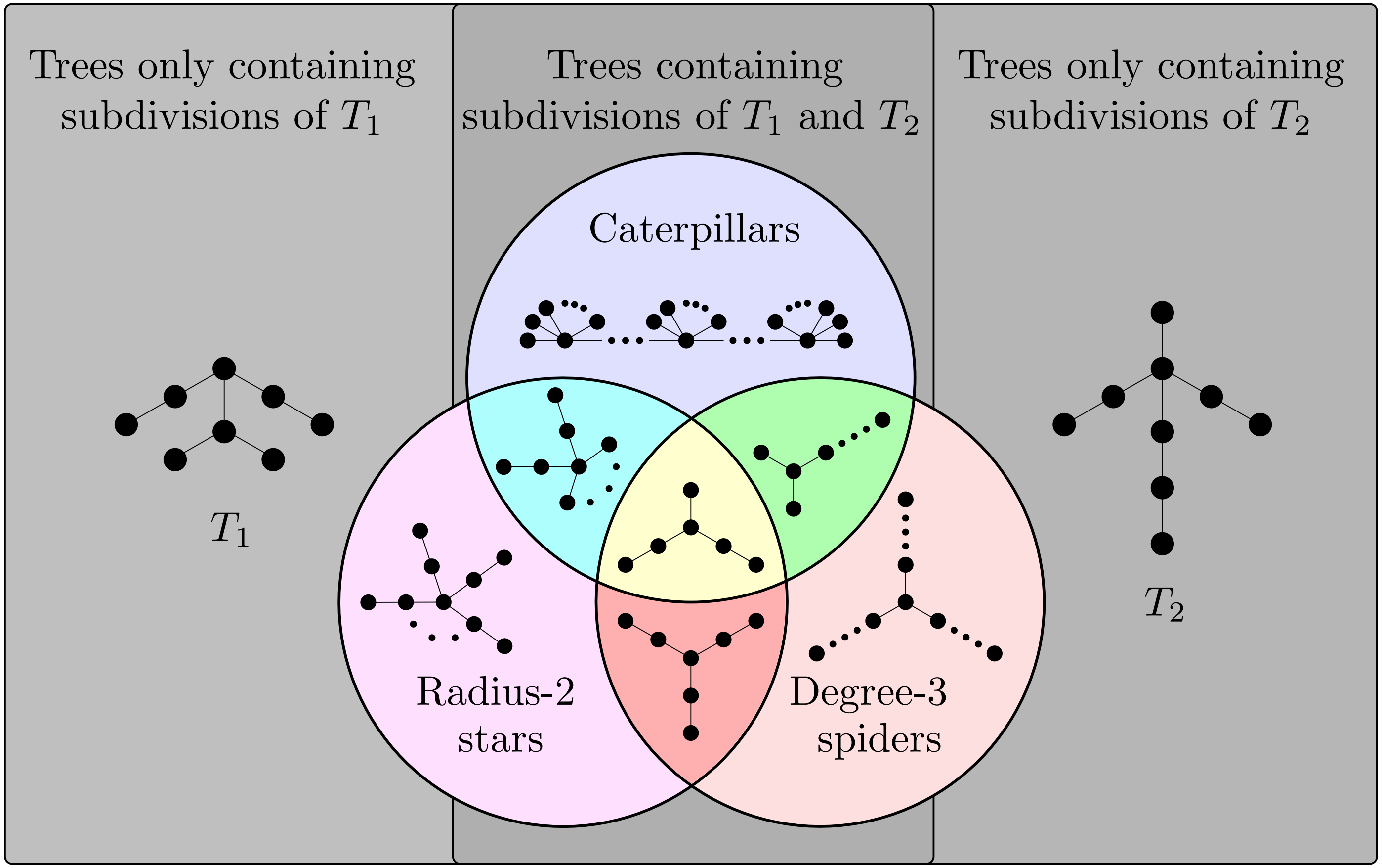

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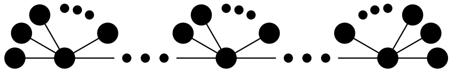

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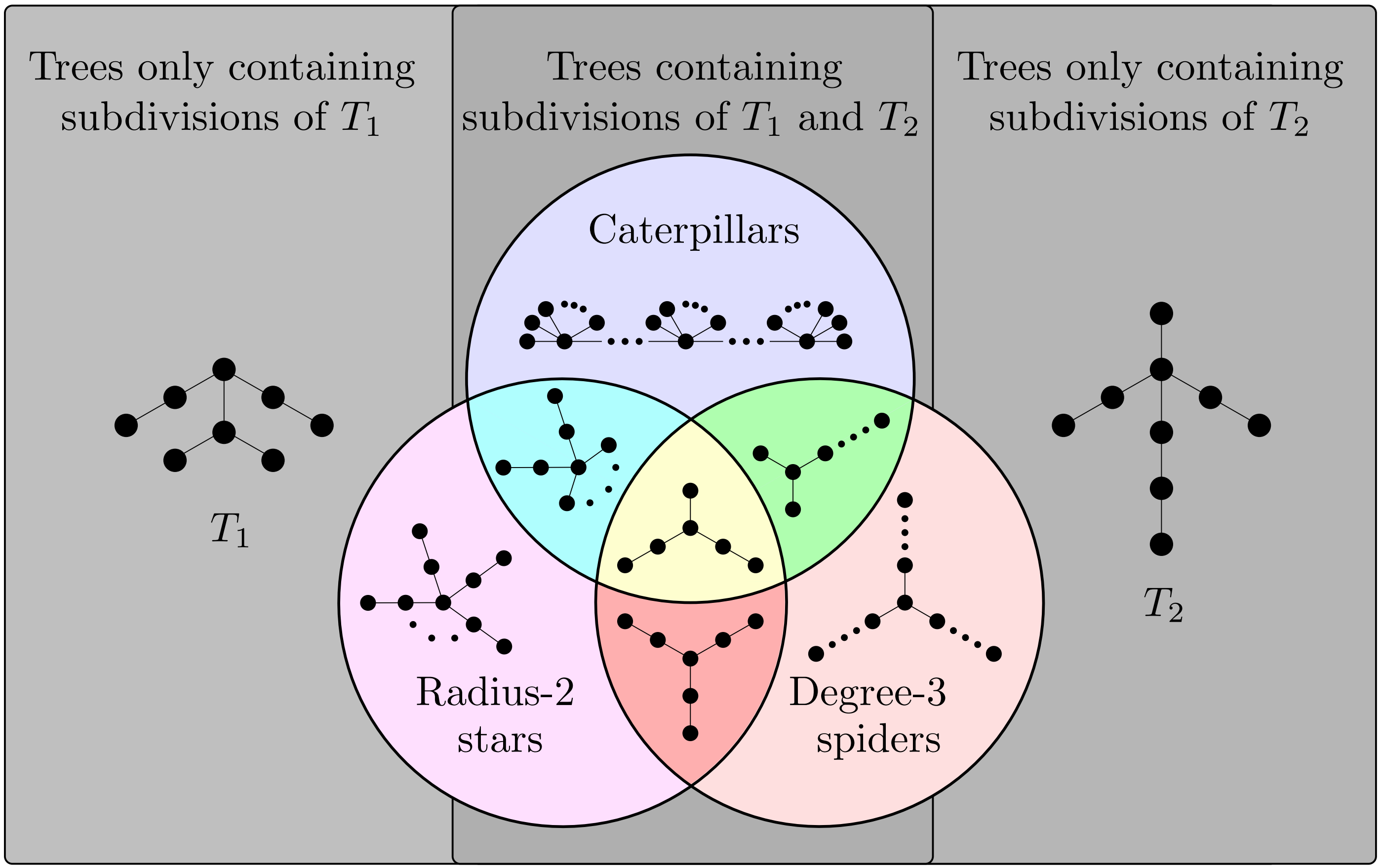

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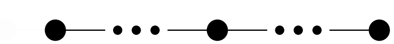

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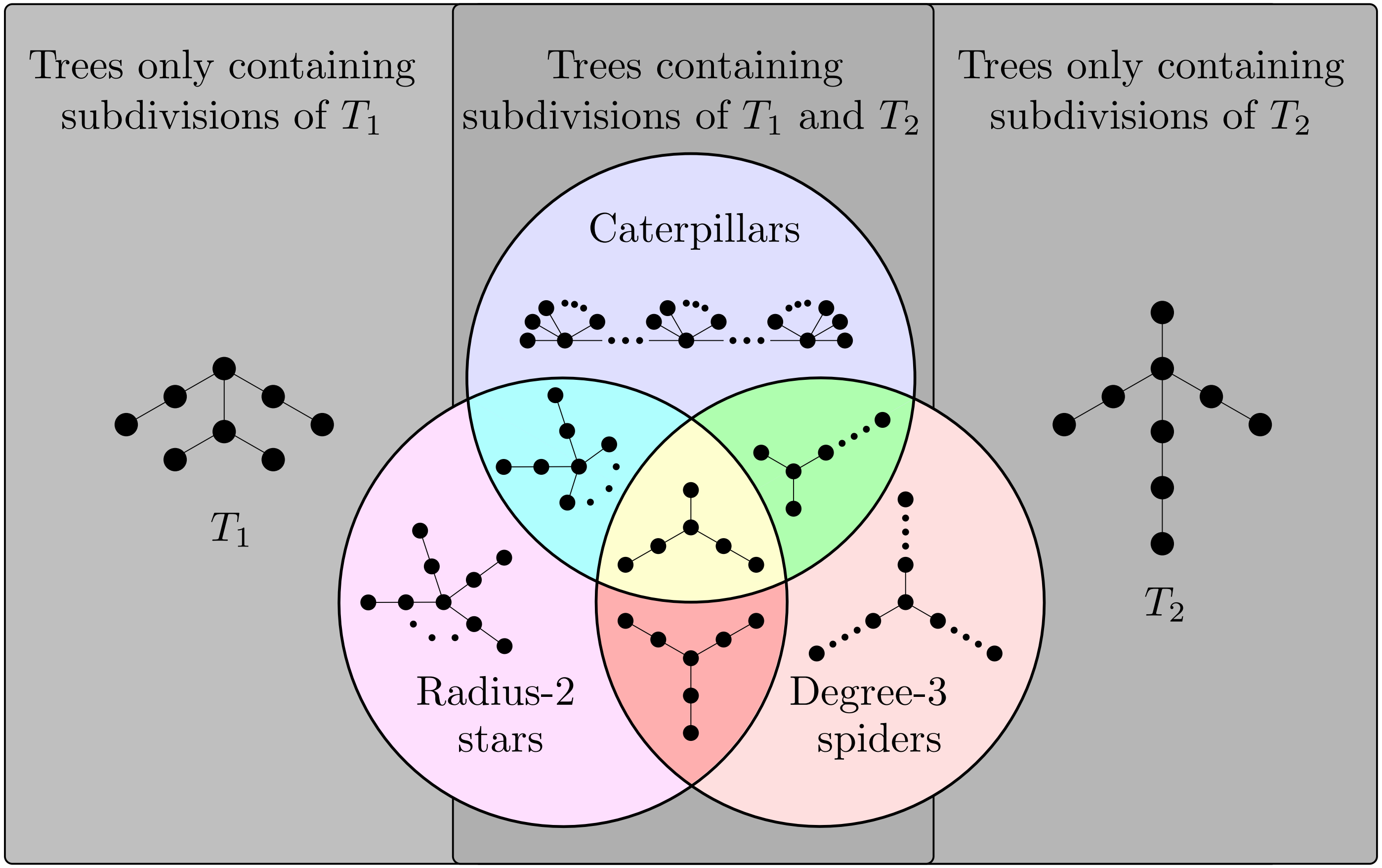

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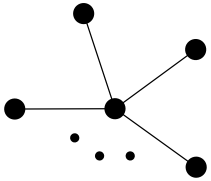

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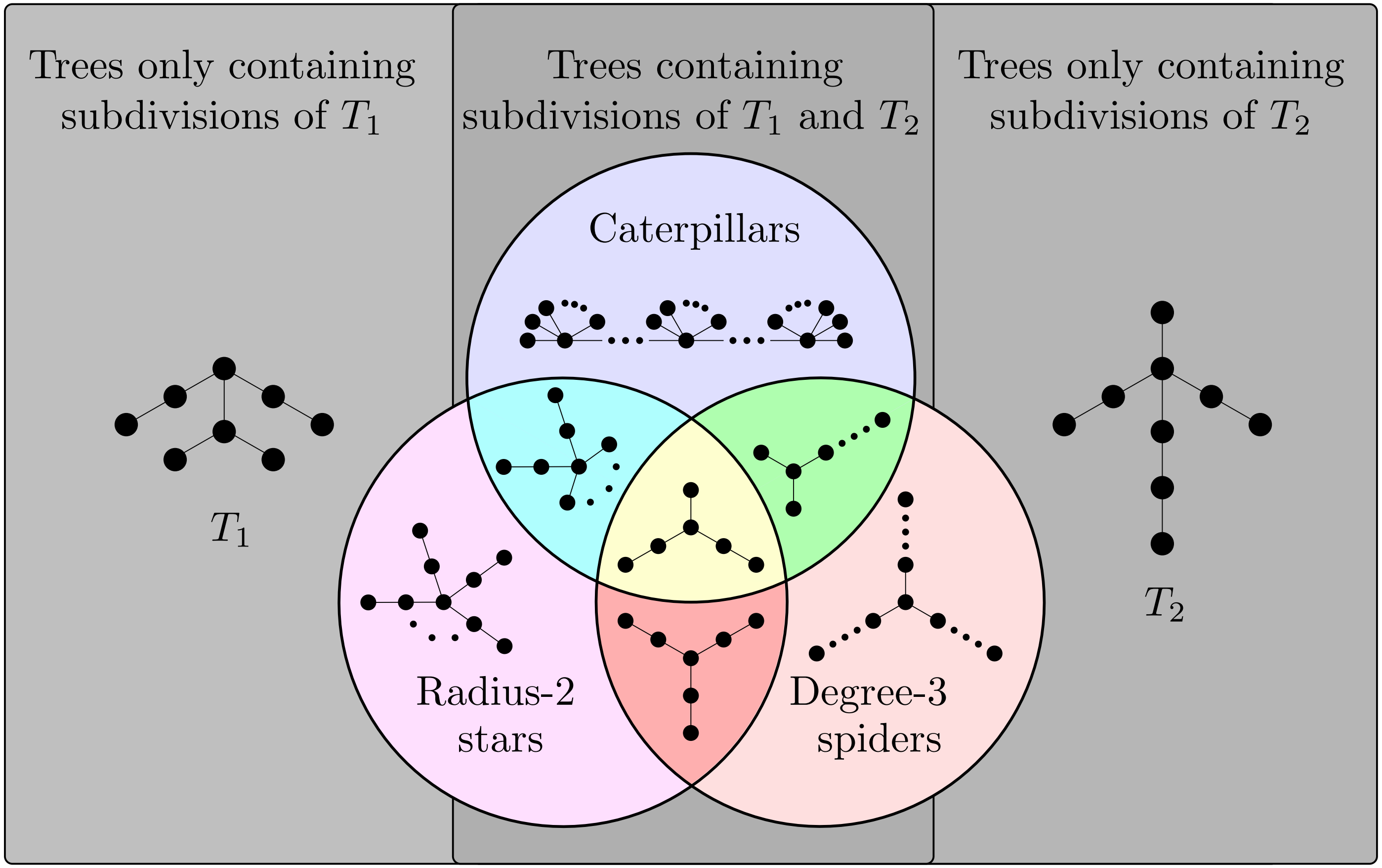

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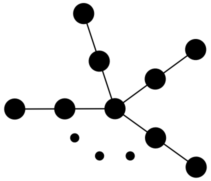

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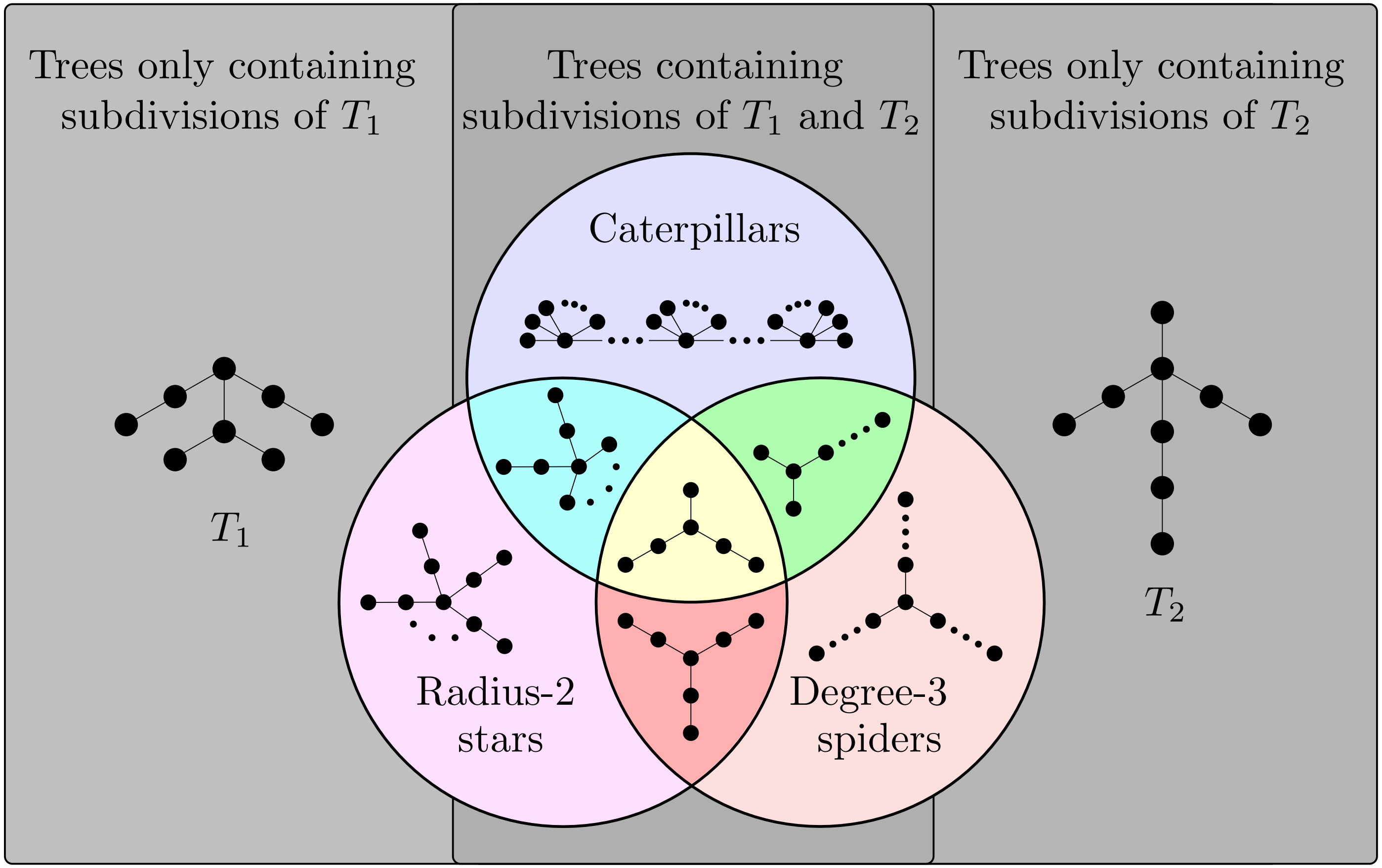

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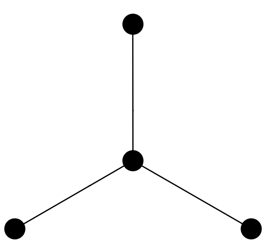

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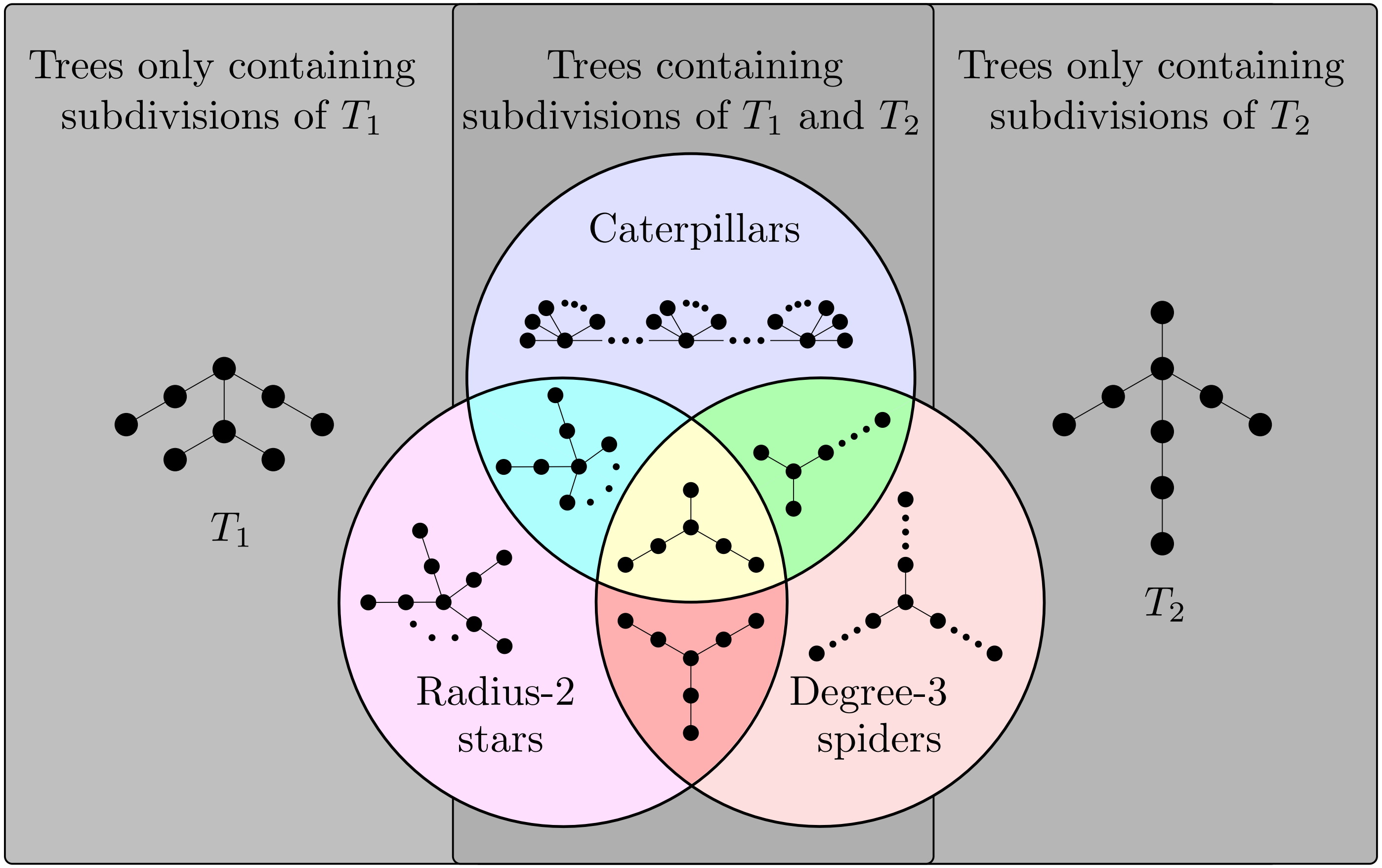

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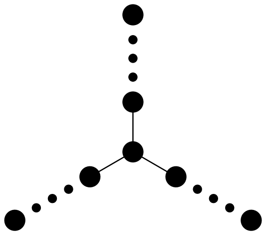

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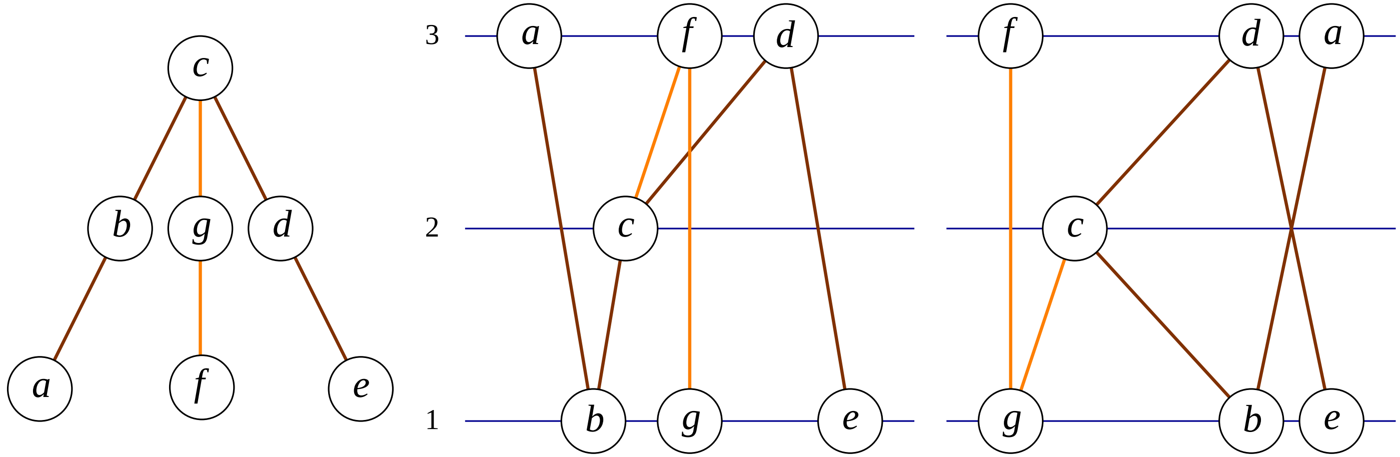
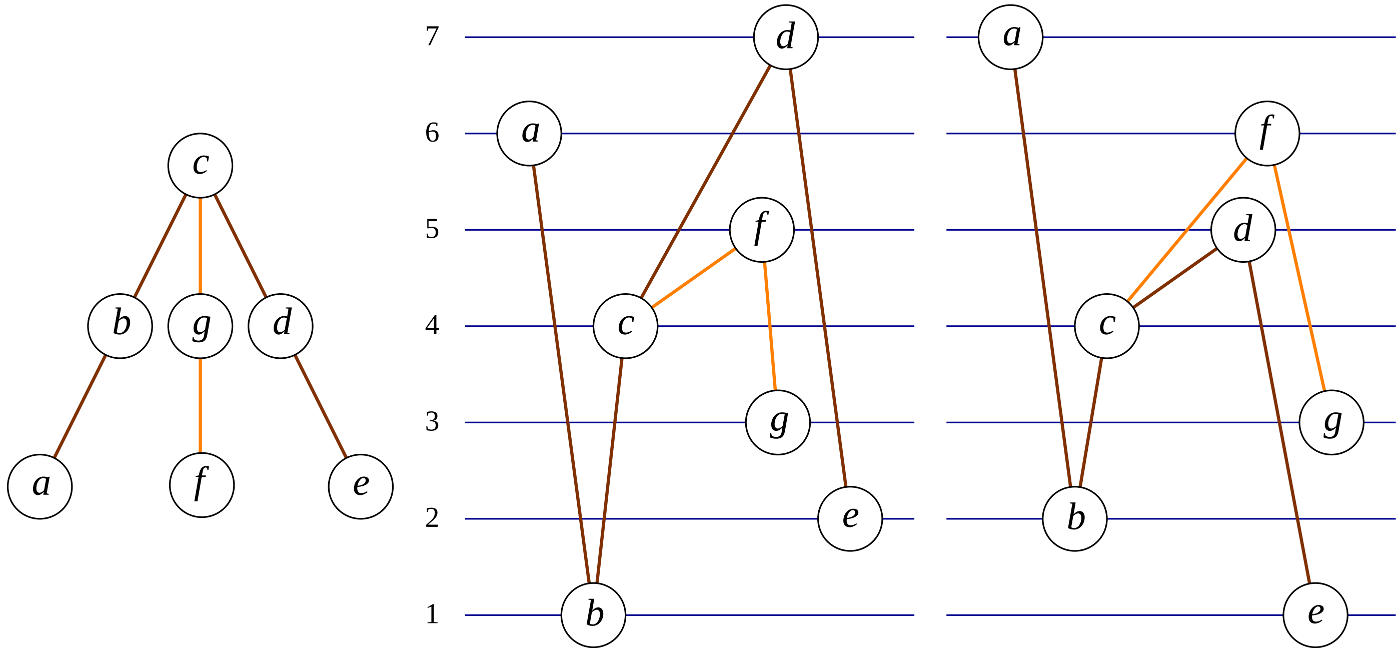
  - But not all planar graphs are level planar for a given level assignment



Level Planarity vs. Standard Planarity

- More restrictive than standard planarity
  - All level planar graphs are planar
  - But not all planar graphs are level planar for a given level assignment
- A ULP graph can have a non-level planar assignment



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    - $u <_Y v_1 - v_2 - \cdots - v_j$

      $\iff u$ **lies below** every point that C shares the same x-coordinate as u

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- Let C be some chain $a{-}b{-}c{-}d{-}e$

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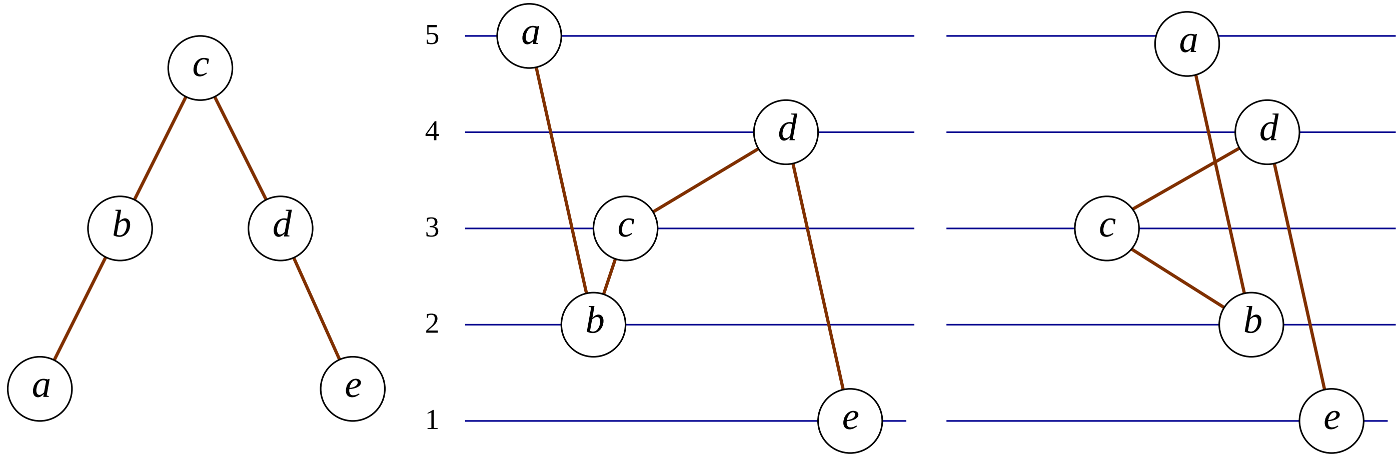

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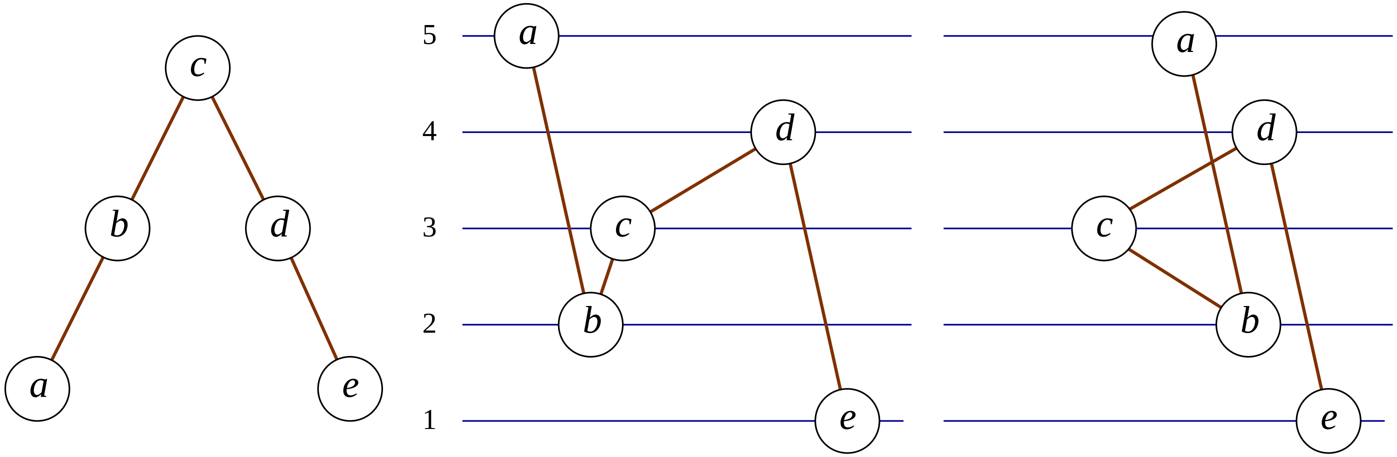

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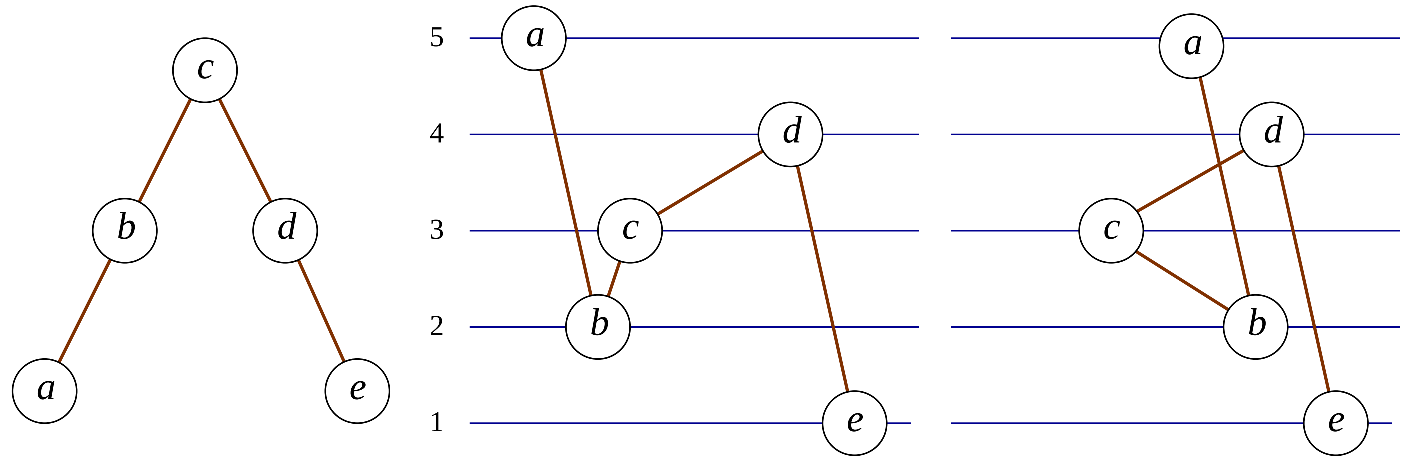

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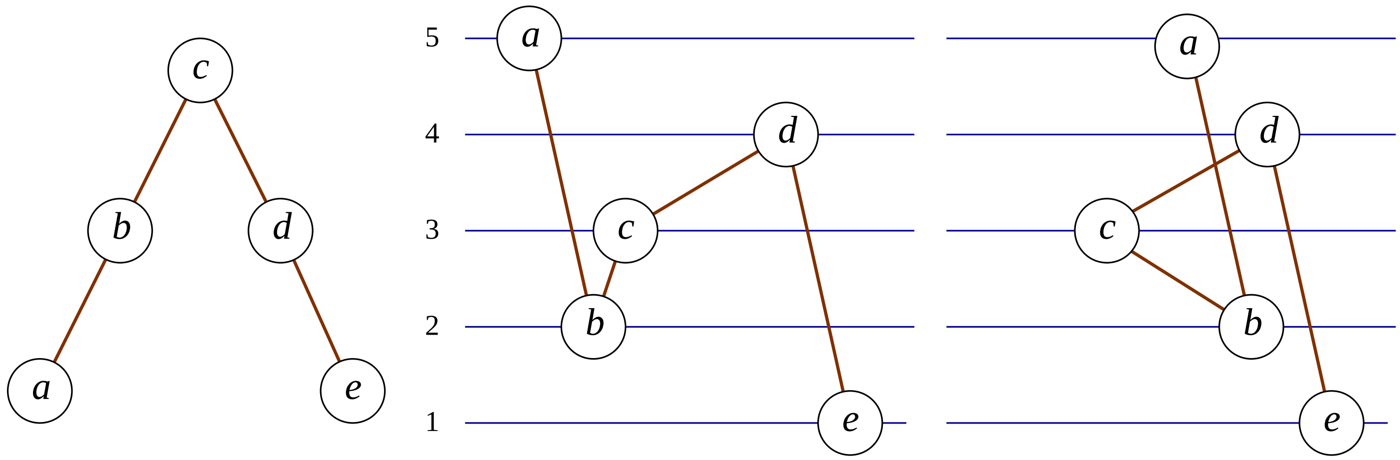

  - Then either
    - $a—b <_X c <_X d—e$ or
    - $d—e <_X c <_X a—b$, i.e, c is between $a—b$ and $d—e$
  - Since otherwise $c—b—a$ will cross $c—d—e$
- So c cannot be leftmost or rightmost without forcing a crossing
  - Can use this property to prove T_1 and T_2 are not ULP

Forbidden Trees – Proof T_1 is not ULP

- Let C be the chain a–b–c–d–e

Forbidden Trees – Proof T_1 is not ULP

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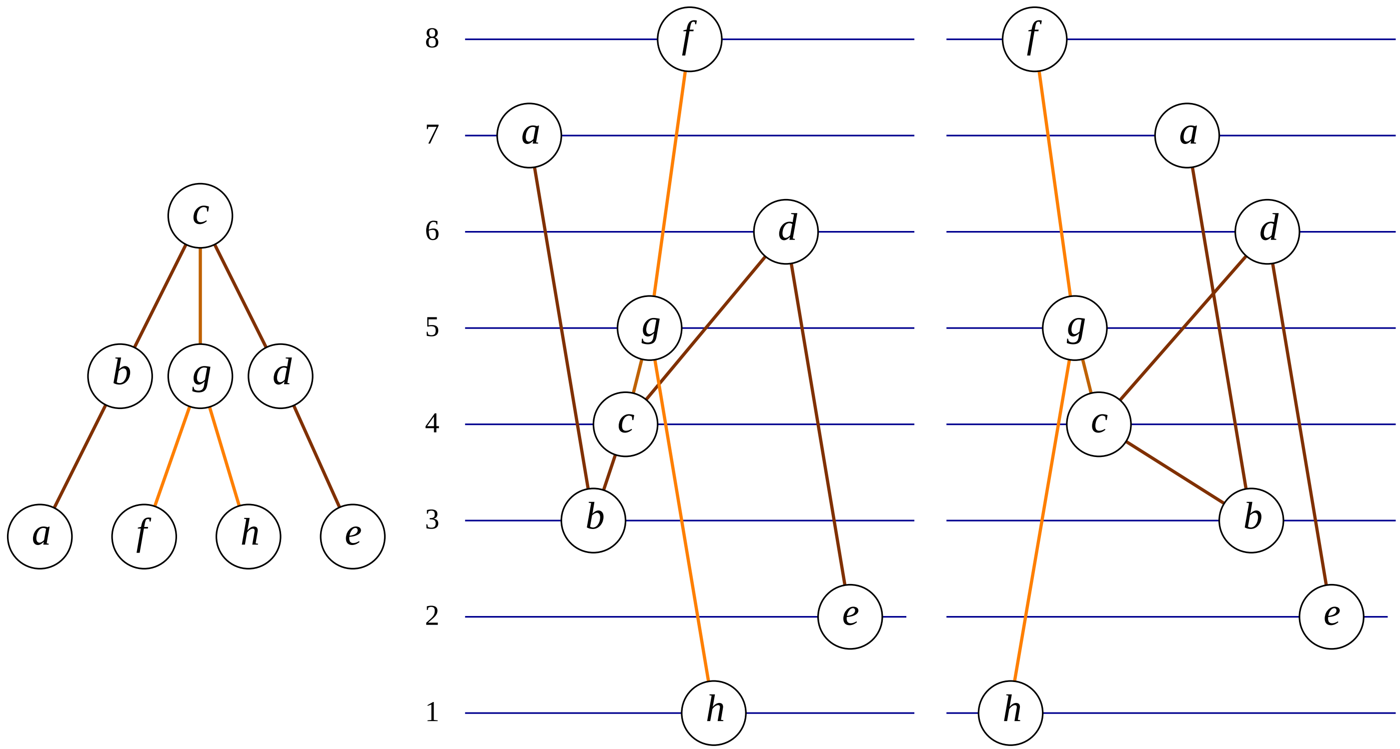
  - Where $\{a, f\} <_Y d <_Y \{c, g\} <_Y b <_Y \{e, h\}$



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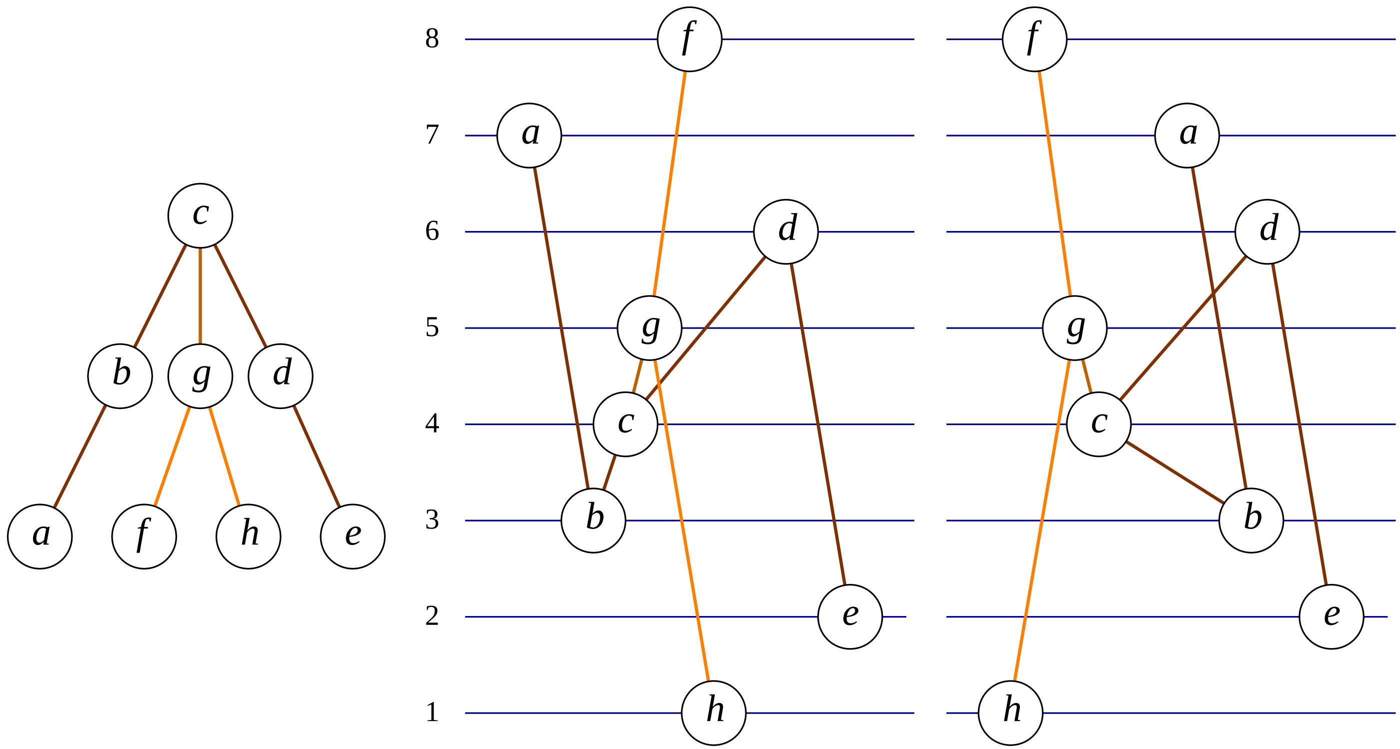

  - Can assume without loss of generality that
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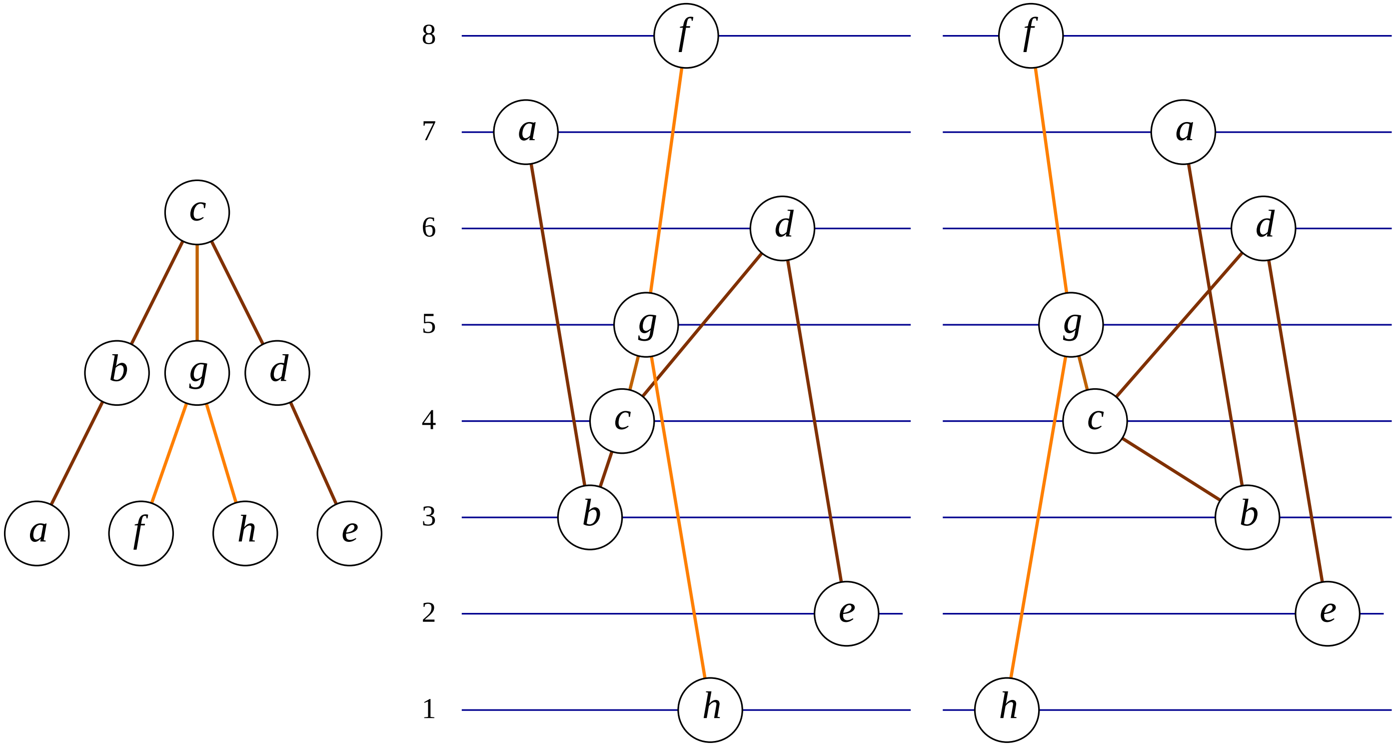

  - Implies that a–$b <_X g <_X d$–e
    - Otherwise, c–g will cross a–b or d–e

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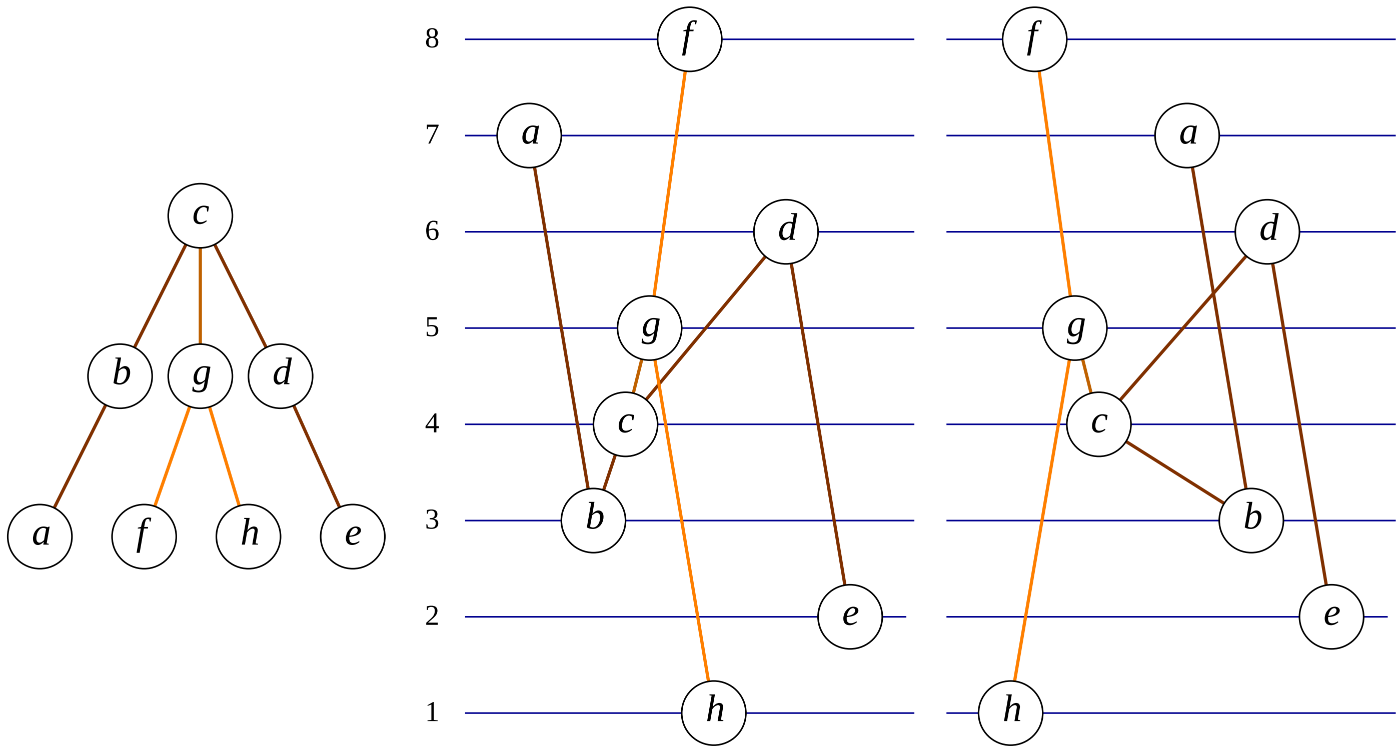

  - Then either $g >_Y a$—b—c—d
    - In which case g—h crosses a—b—c—d

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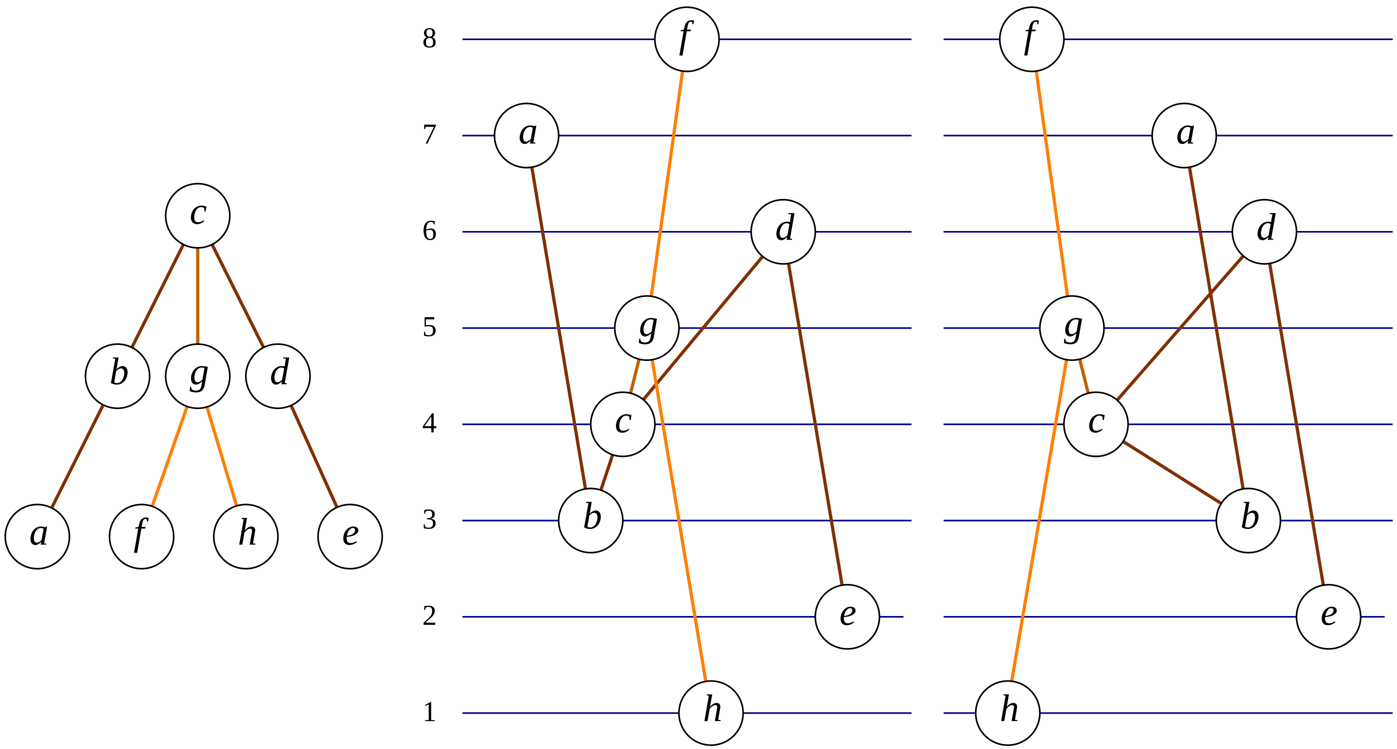

  - Or $g <_Y b$–c–d–e
    - In which case g–f crosses b–c–d–e

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- Let C be the chain a–b–c–d–e
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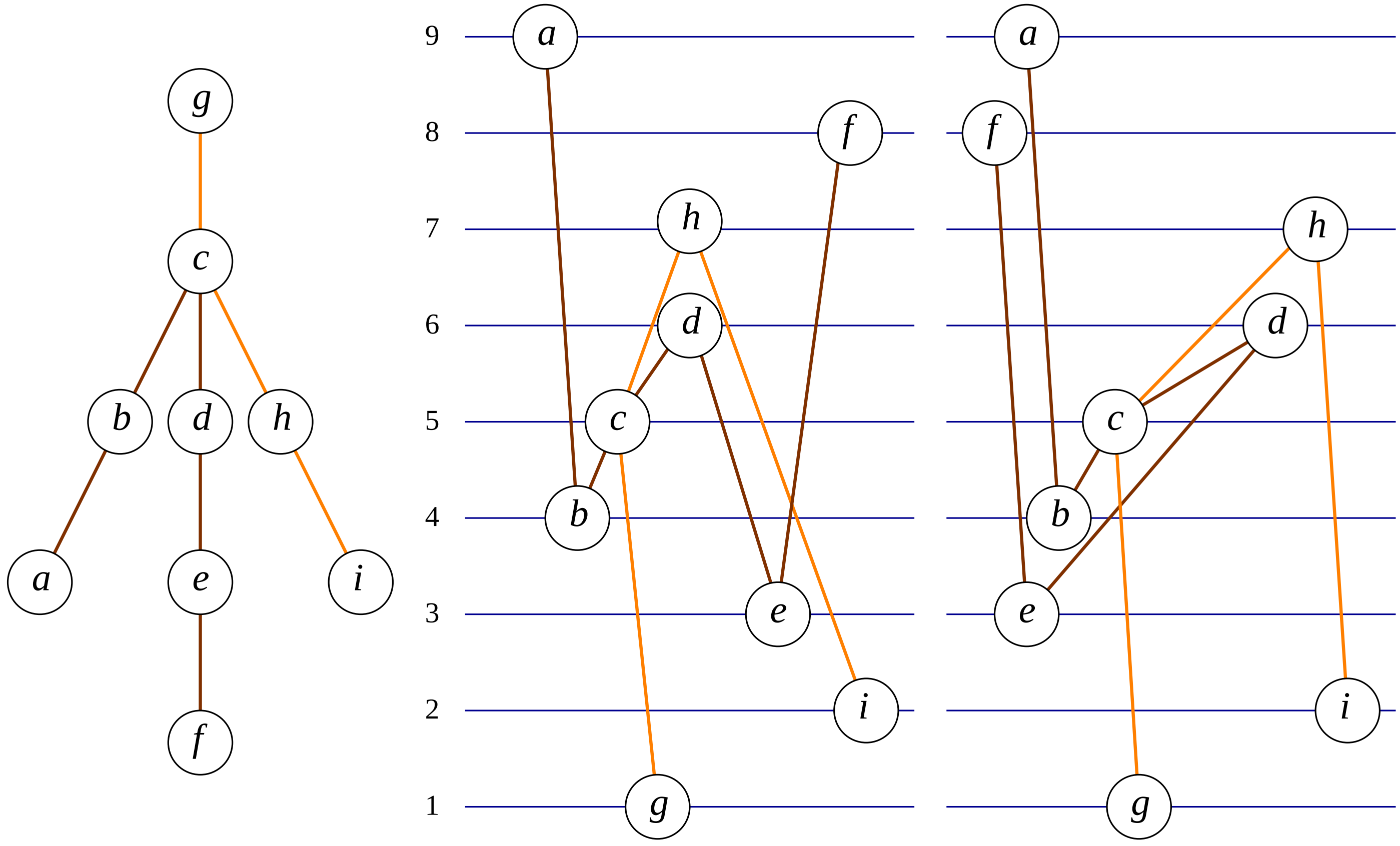

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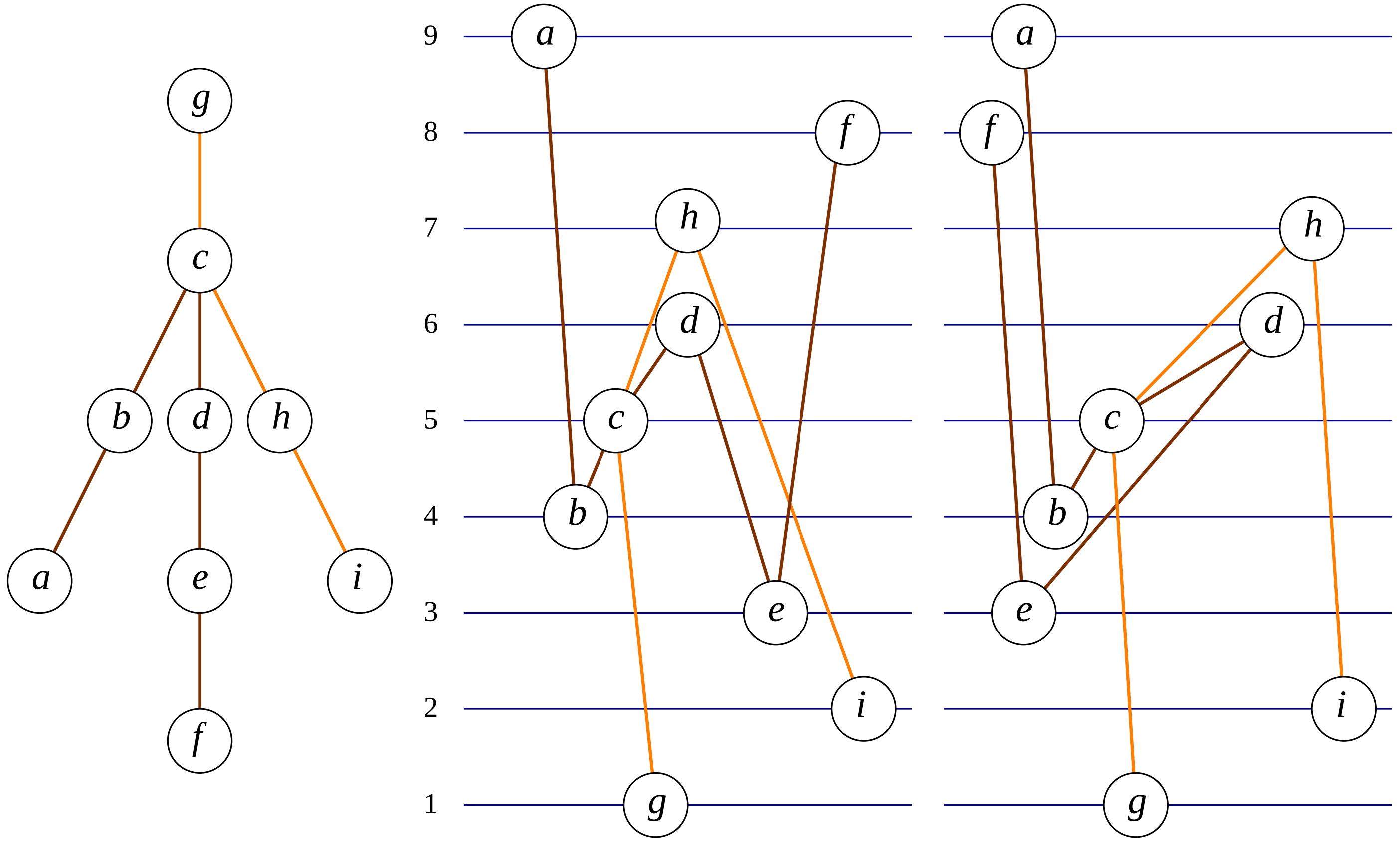

  - One can assume without loss of generality that
    - $a–b <_X c <_X d–e$ since $a <_Y d <_Y c <_Y b <_Y e$

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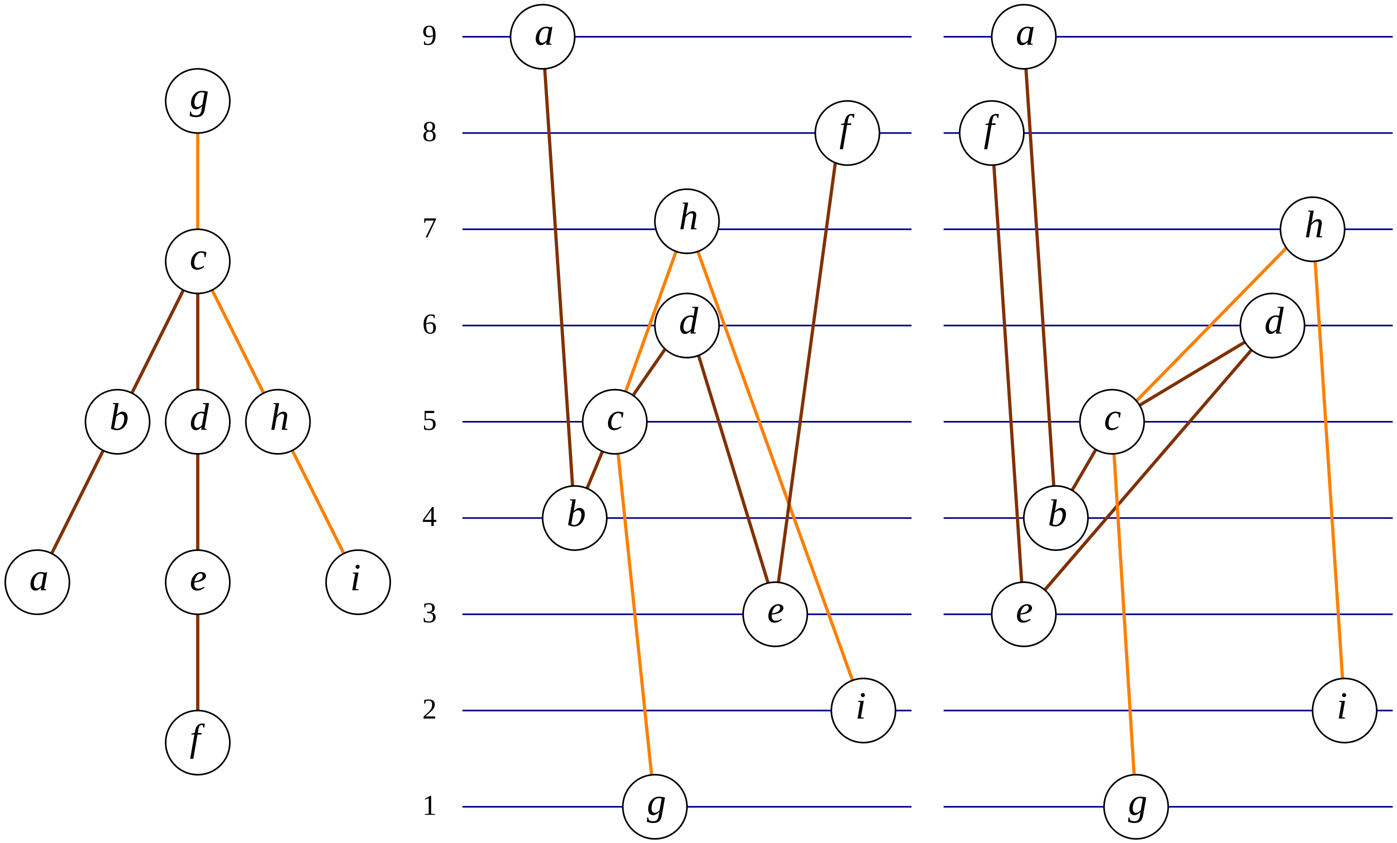

  - AND one can also assume that
    - a–$b <_X c <_X h$–i since $a <_Y h <_Y c <_Y b <_Y i$

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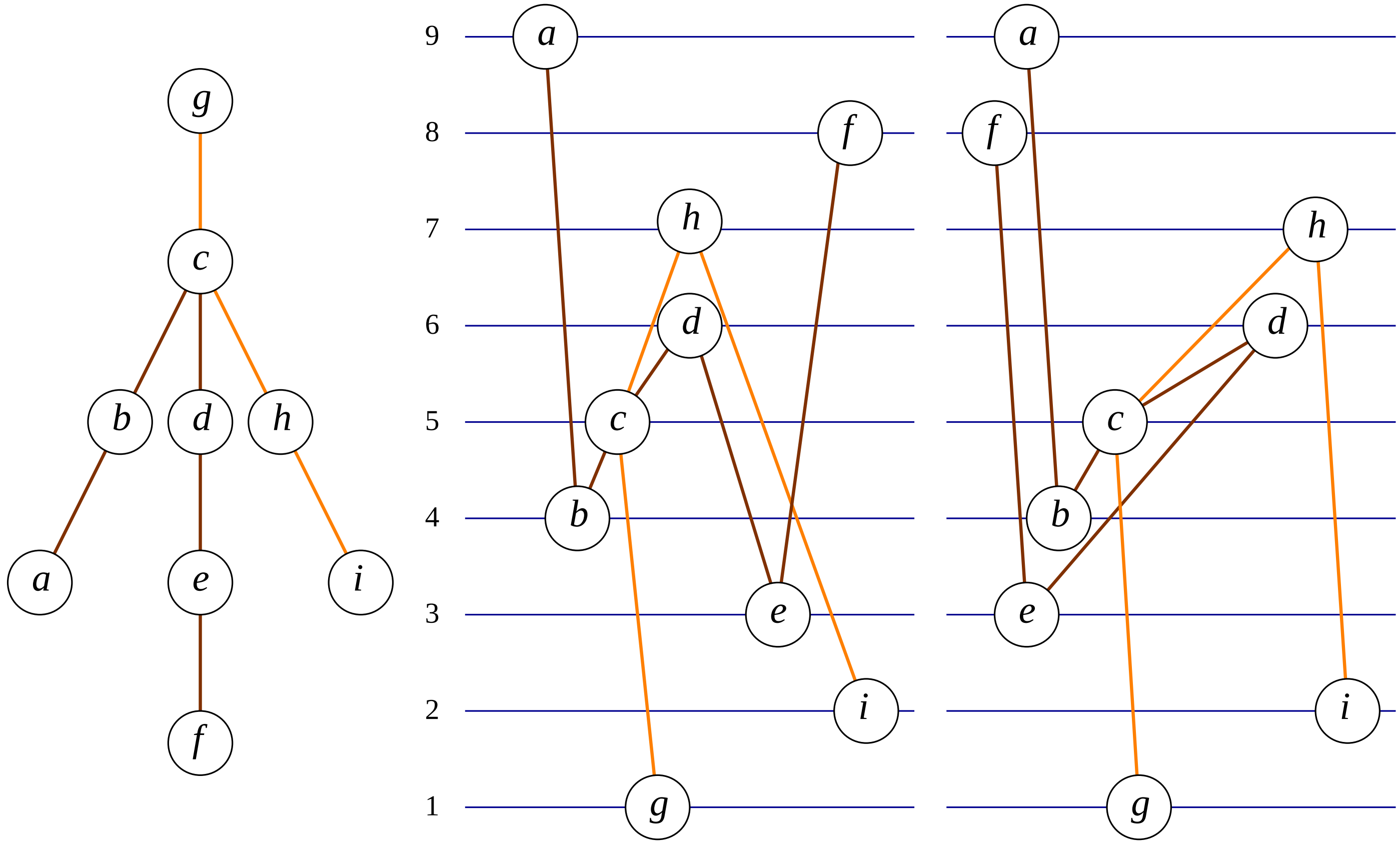

  - Implies that c–$g <_X e <_X h$–i
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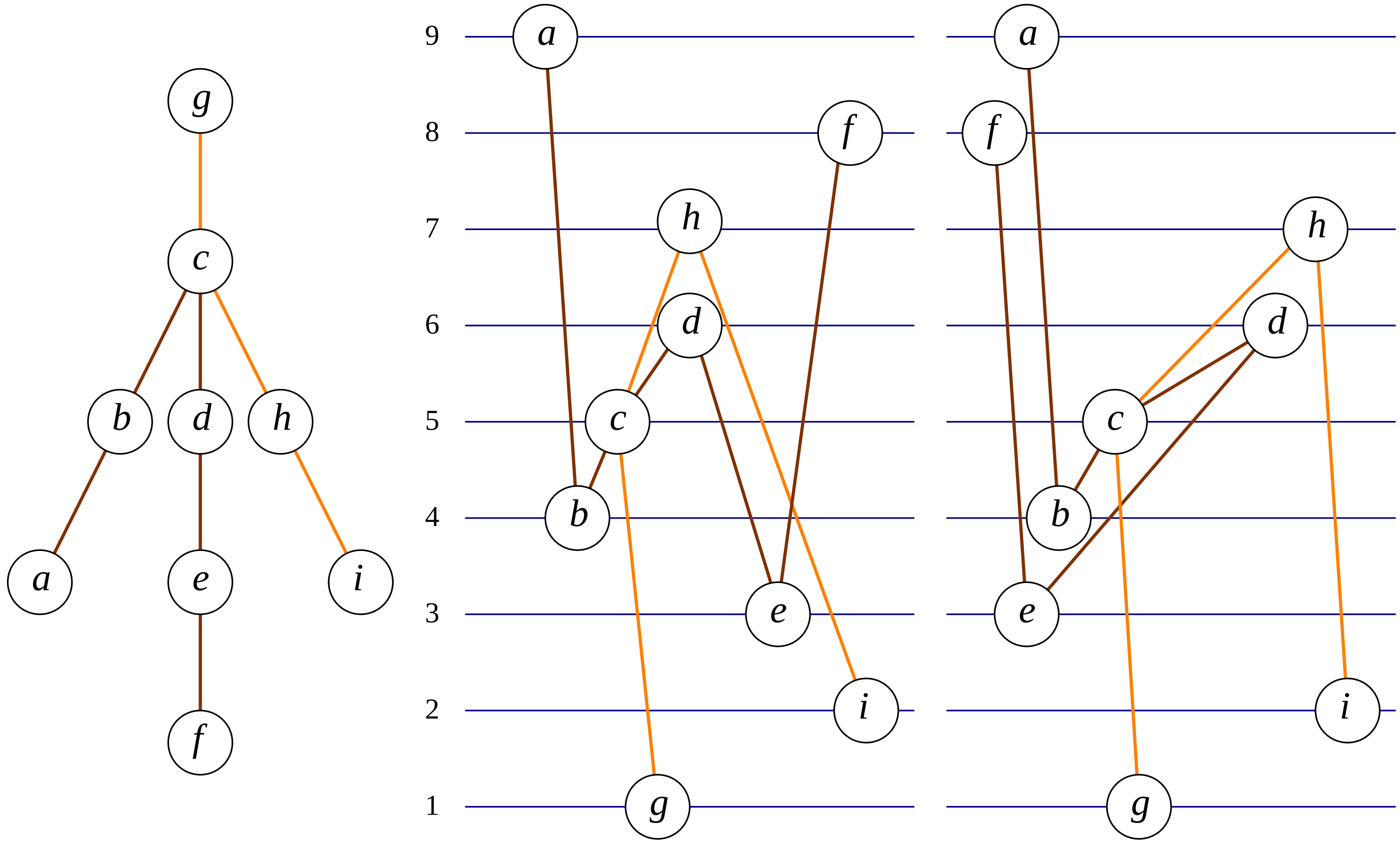

  - However, this implies that $e <_Y g{-}c{-}h{-}i$
    - In which case $e{-}f$ crosses $g{-}c{-}h{-}i$

Forbidden Trees – Lemma and Corollary

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- Proof idea:
  - Assign intermediate levels to vertices of subdivided edges

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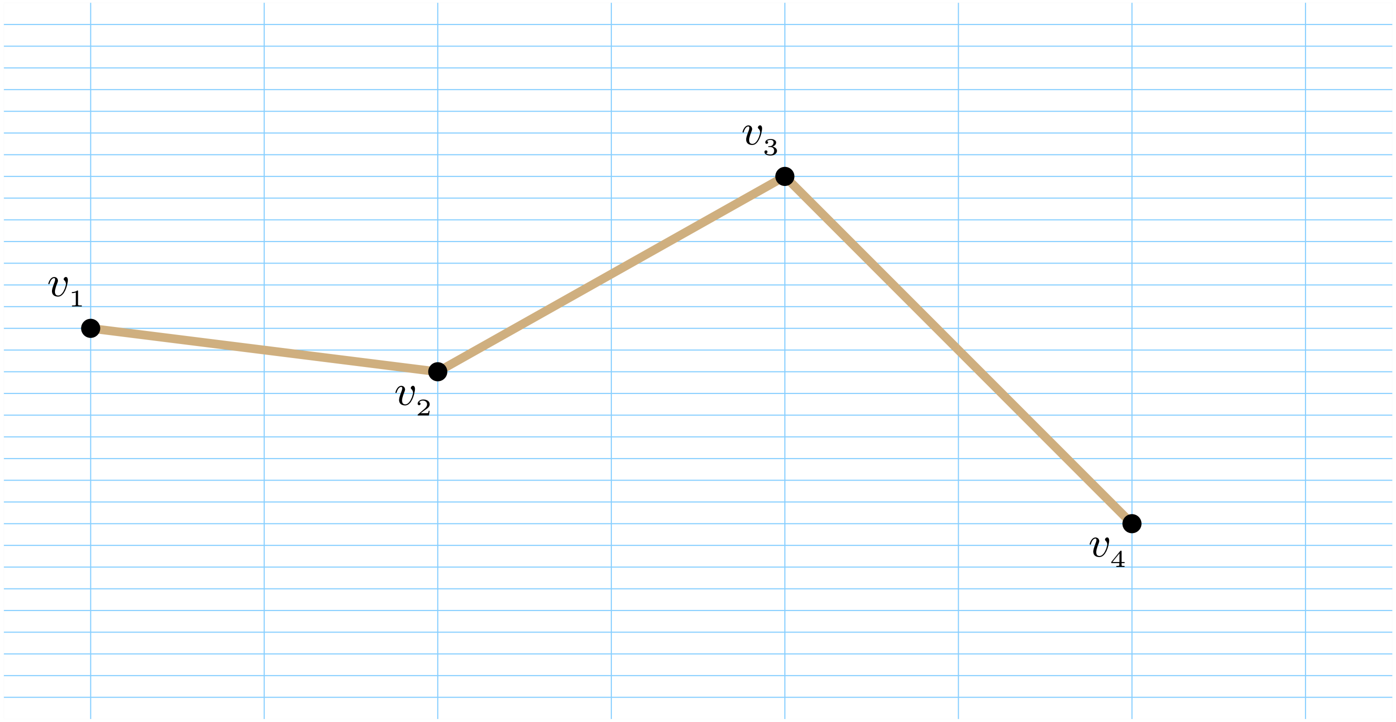

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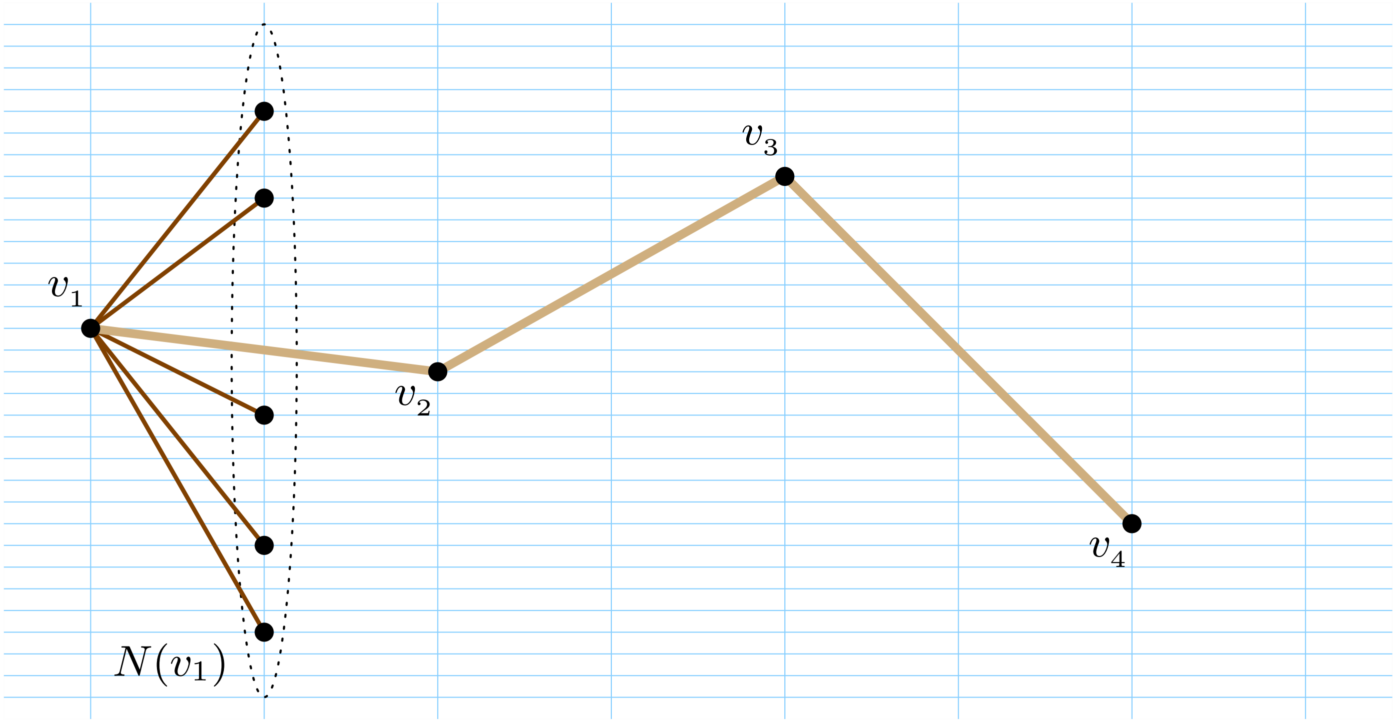

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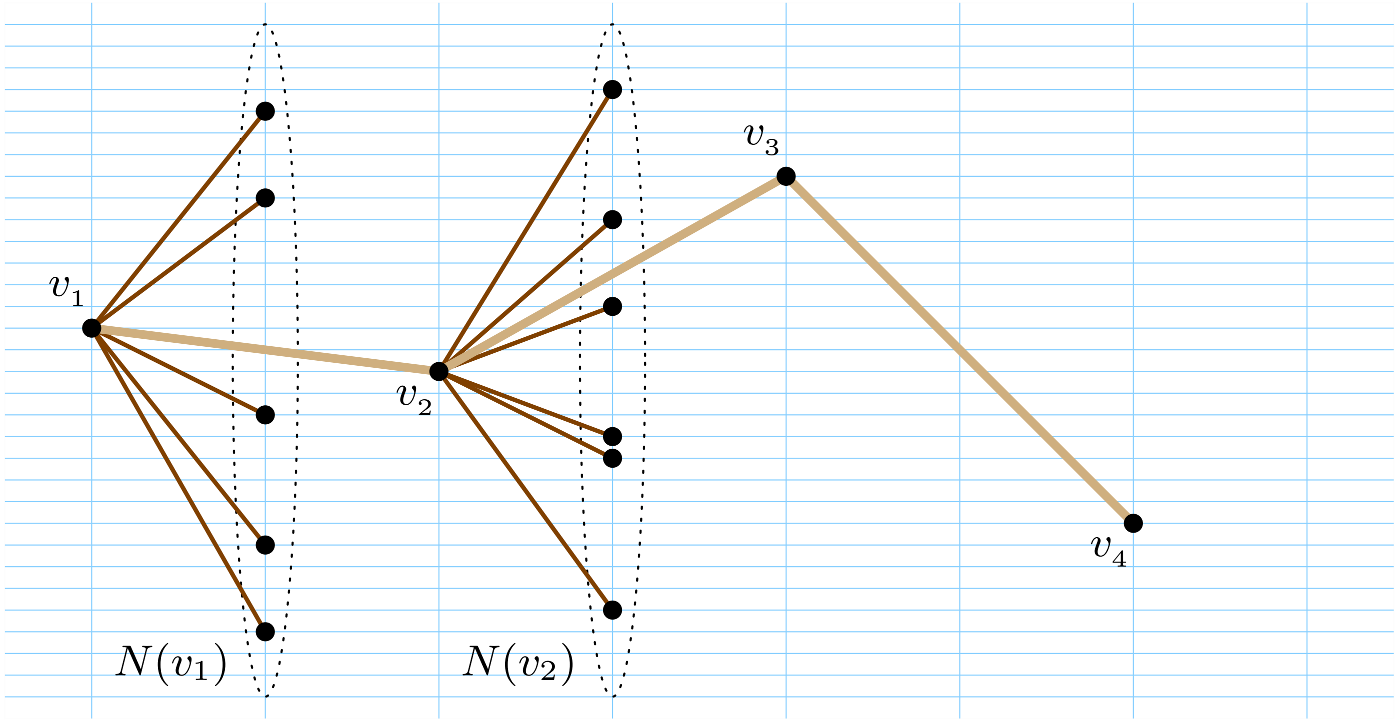

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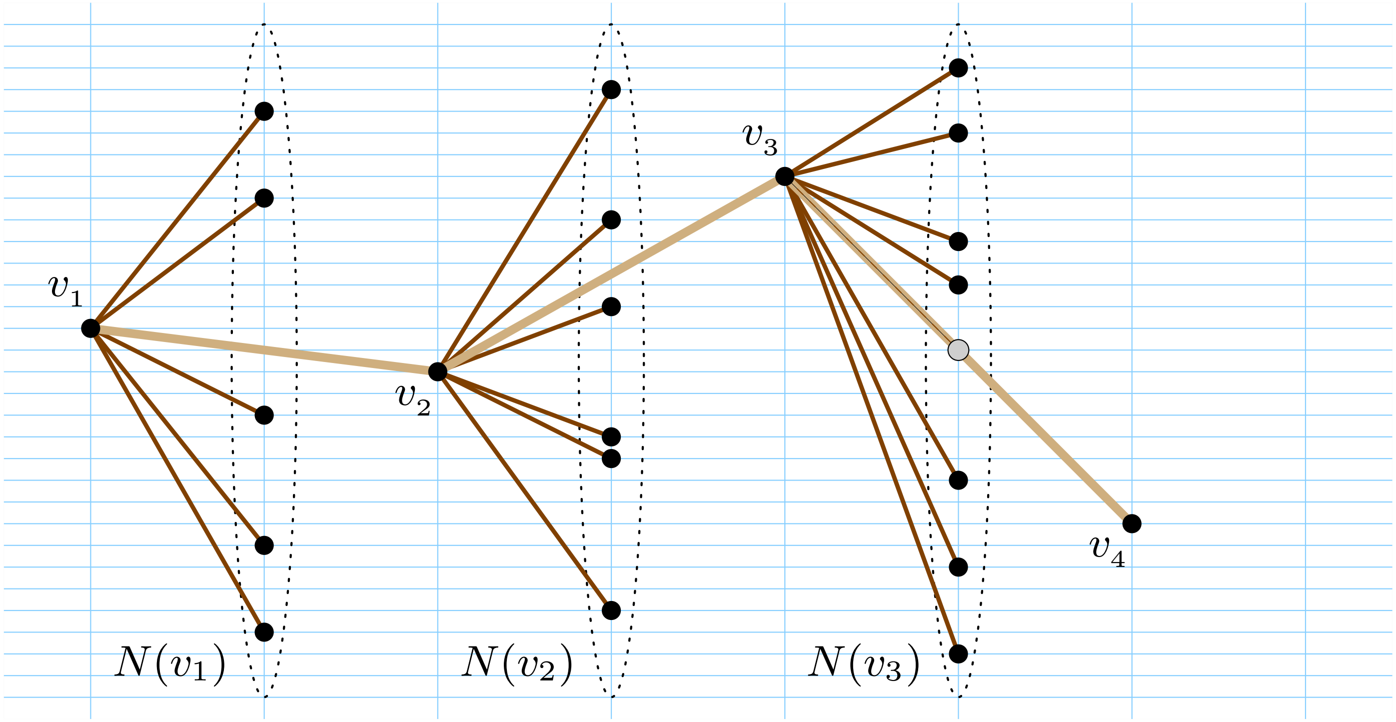

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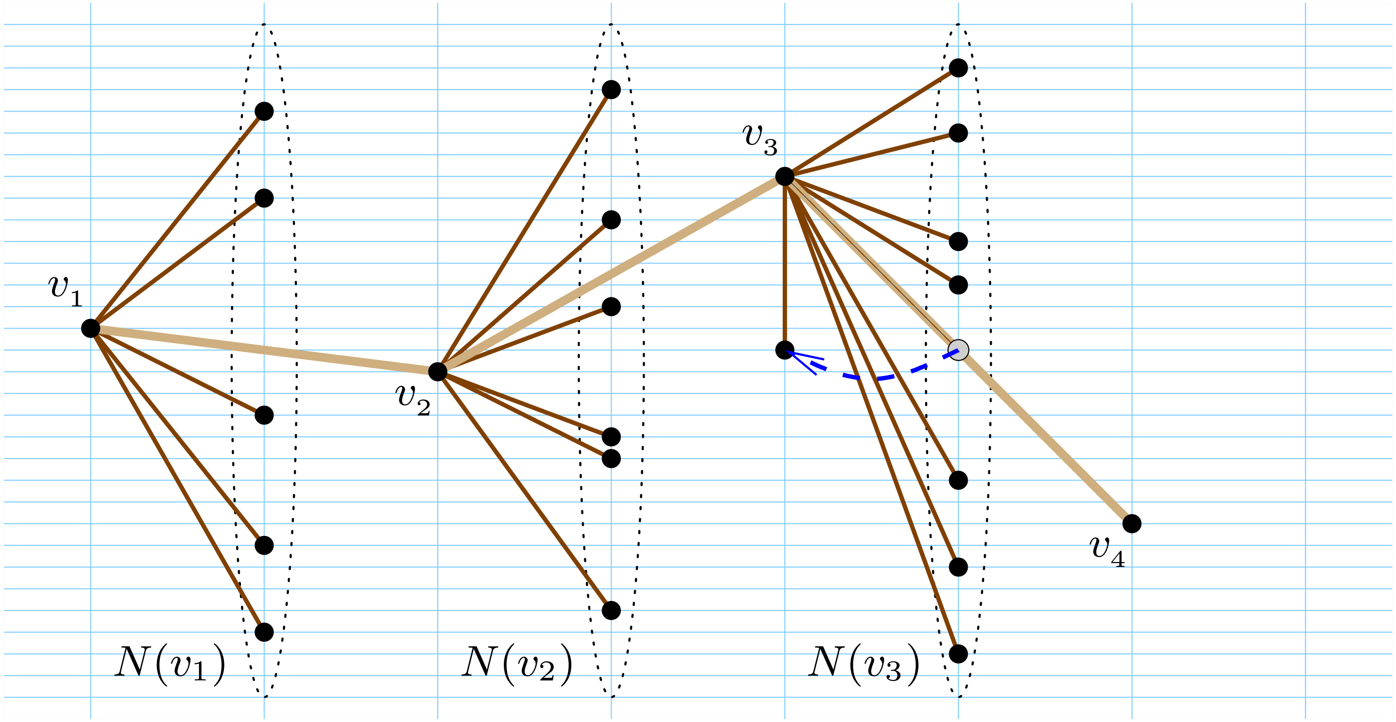

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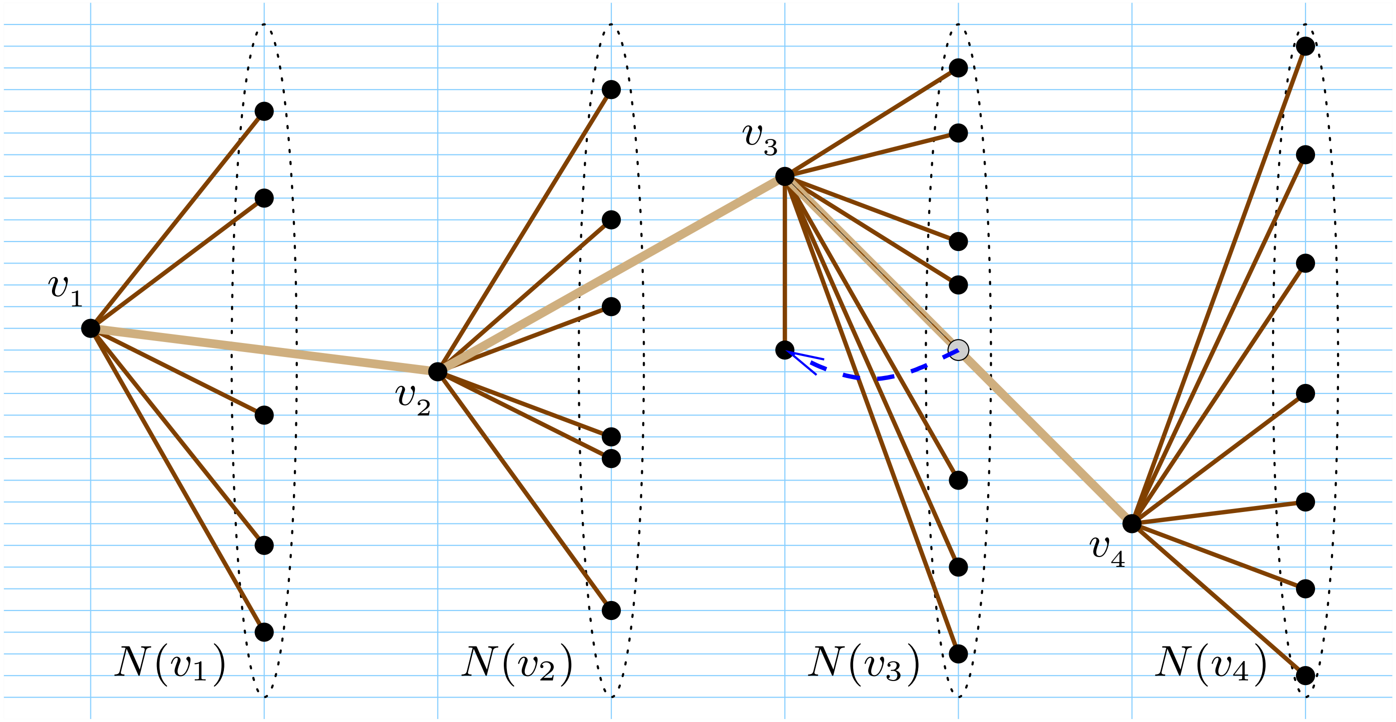

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Lemma 4 *An n-vertex radius-2 star $T(V, E)$ can be n-level realized in $O(n)$ time on a $(2n+3) \times n$ grid for any vertex labeling $\phi : V \xrightarrow[\text{onto}]{1:1} \{1, 2, \ldots, n\}$.*

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- Proof idea:
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    - Use imaginary levels above and below to determine x-coordinate

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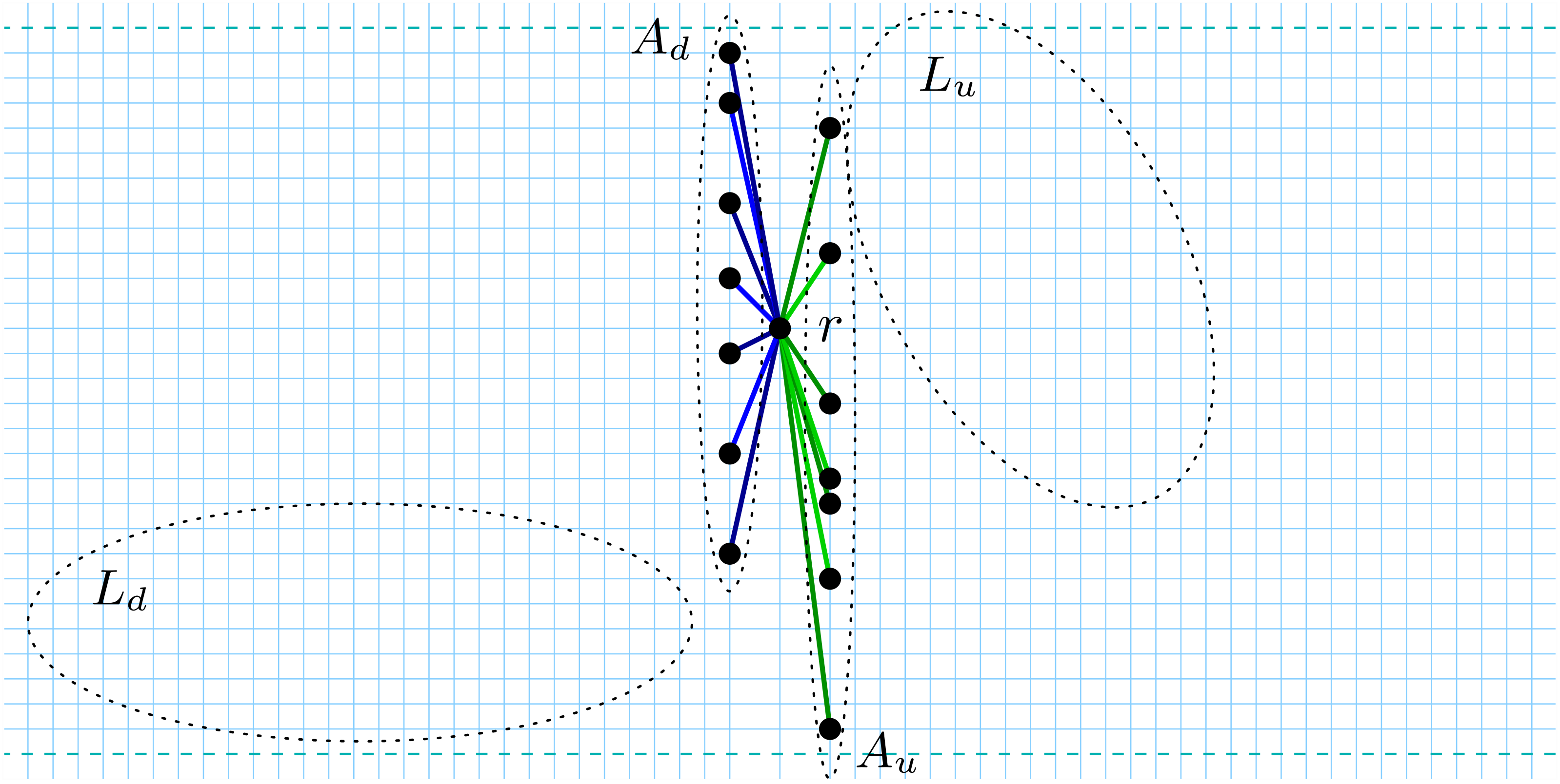

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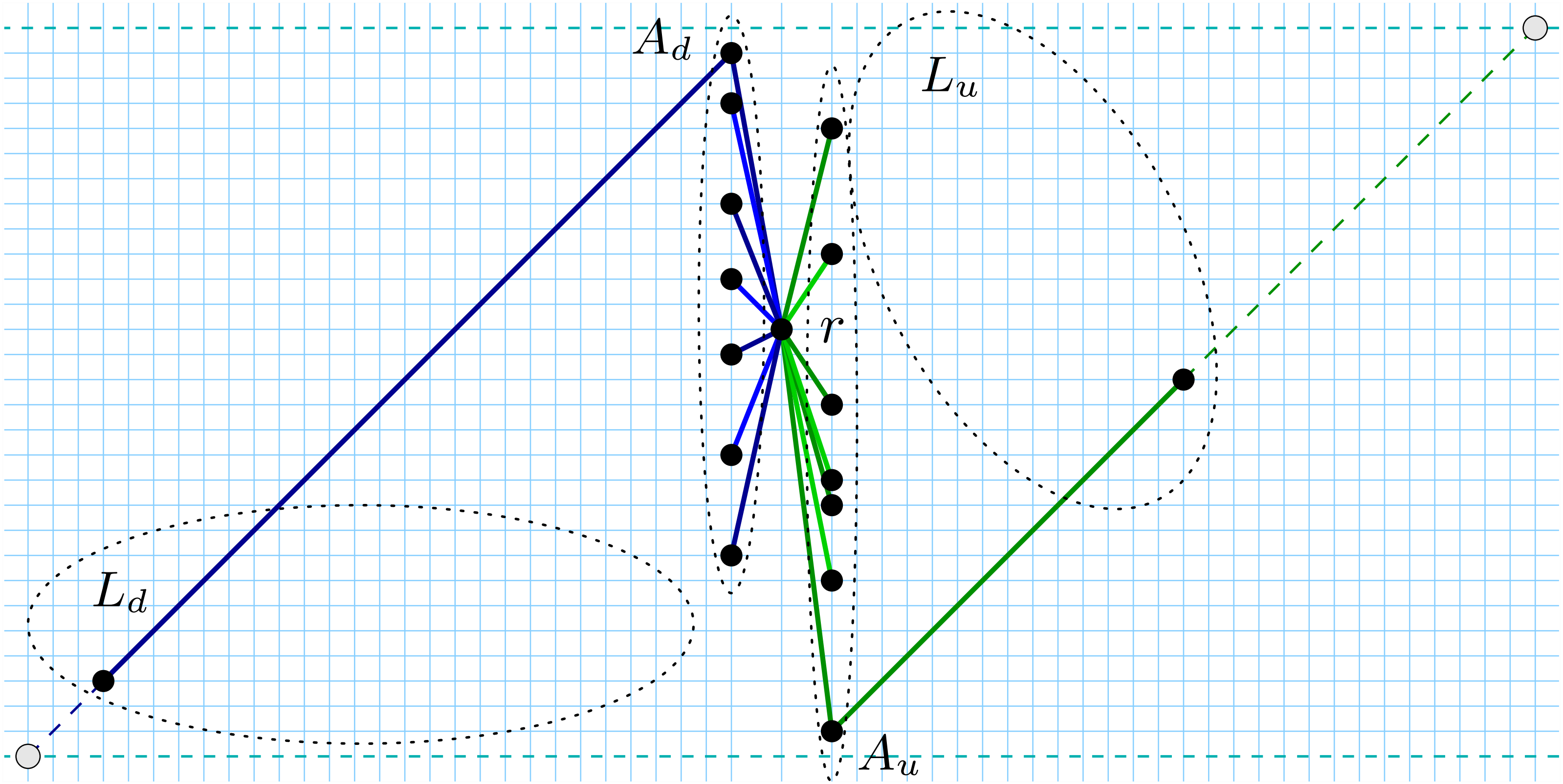

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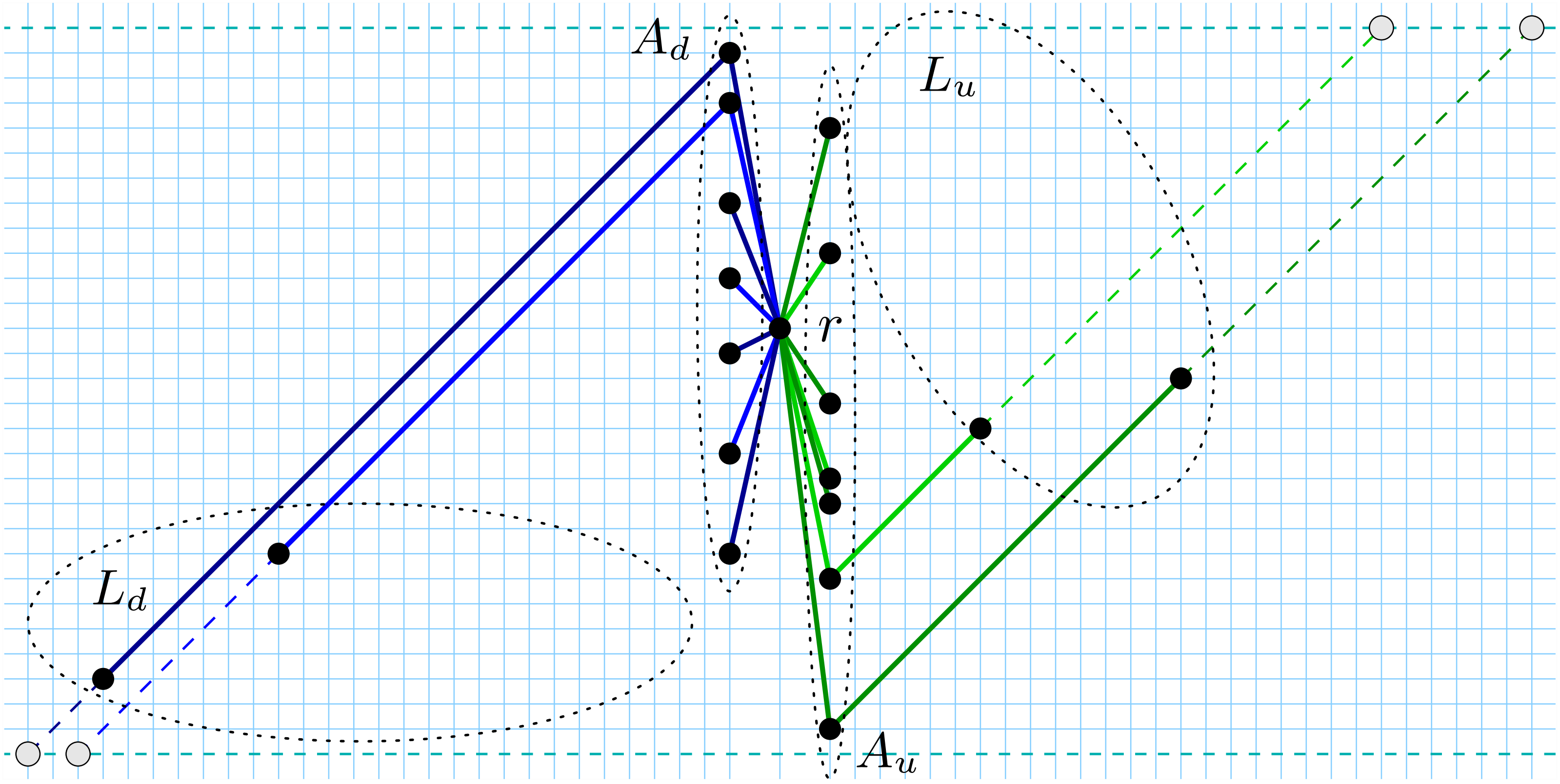

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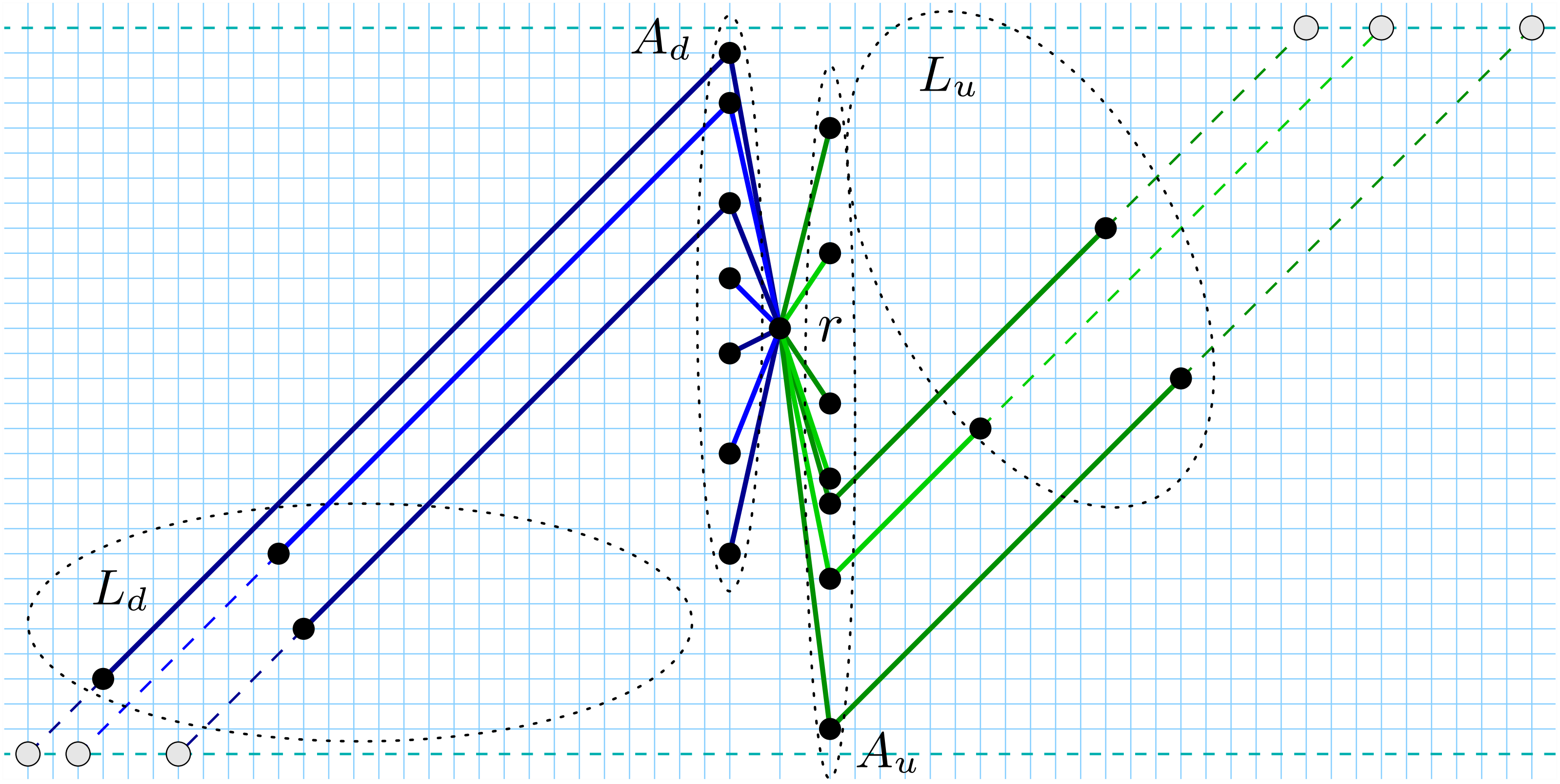

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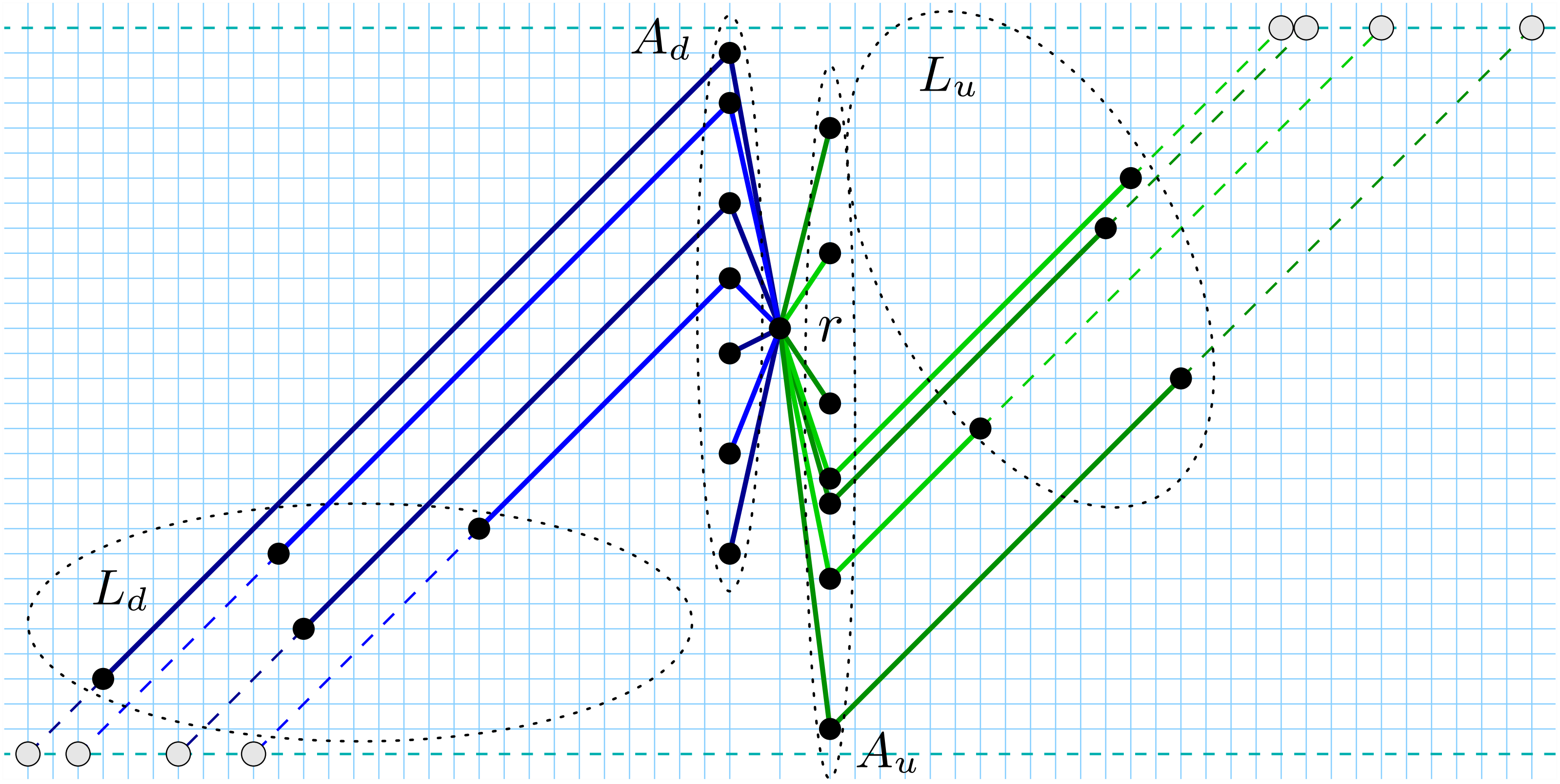

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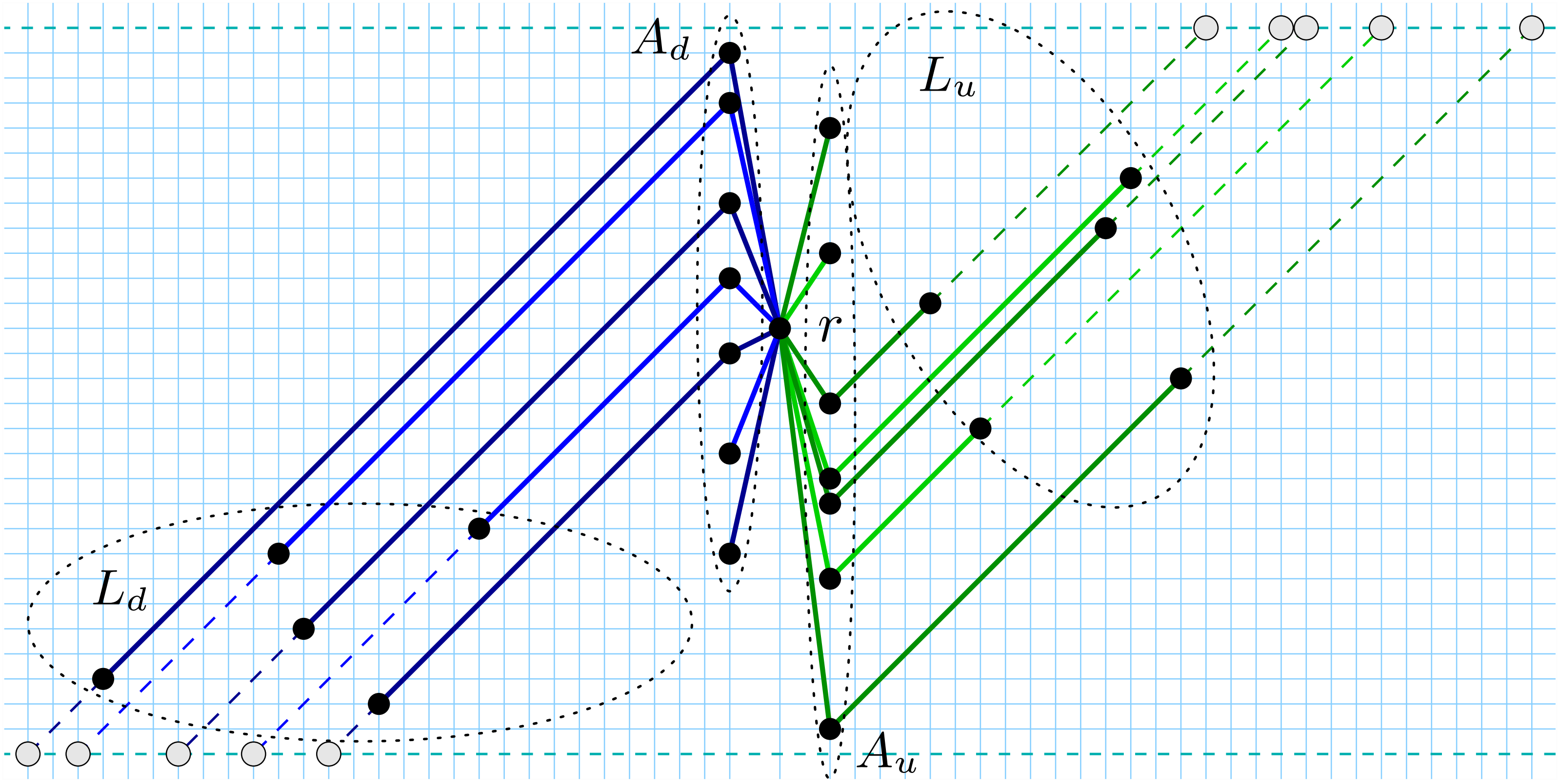

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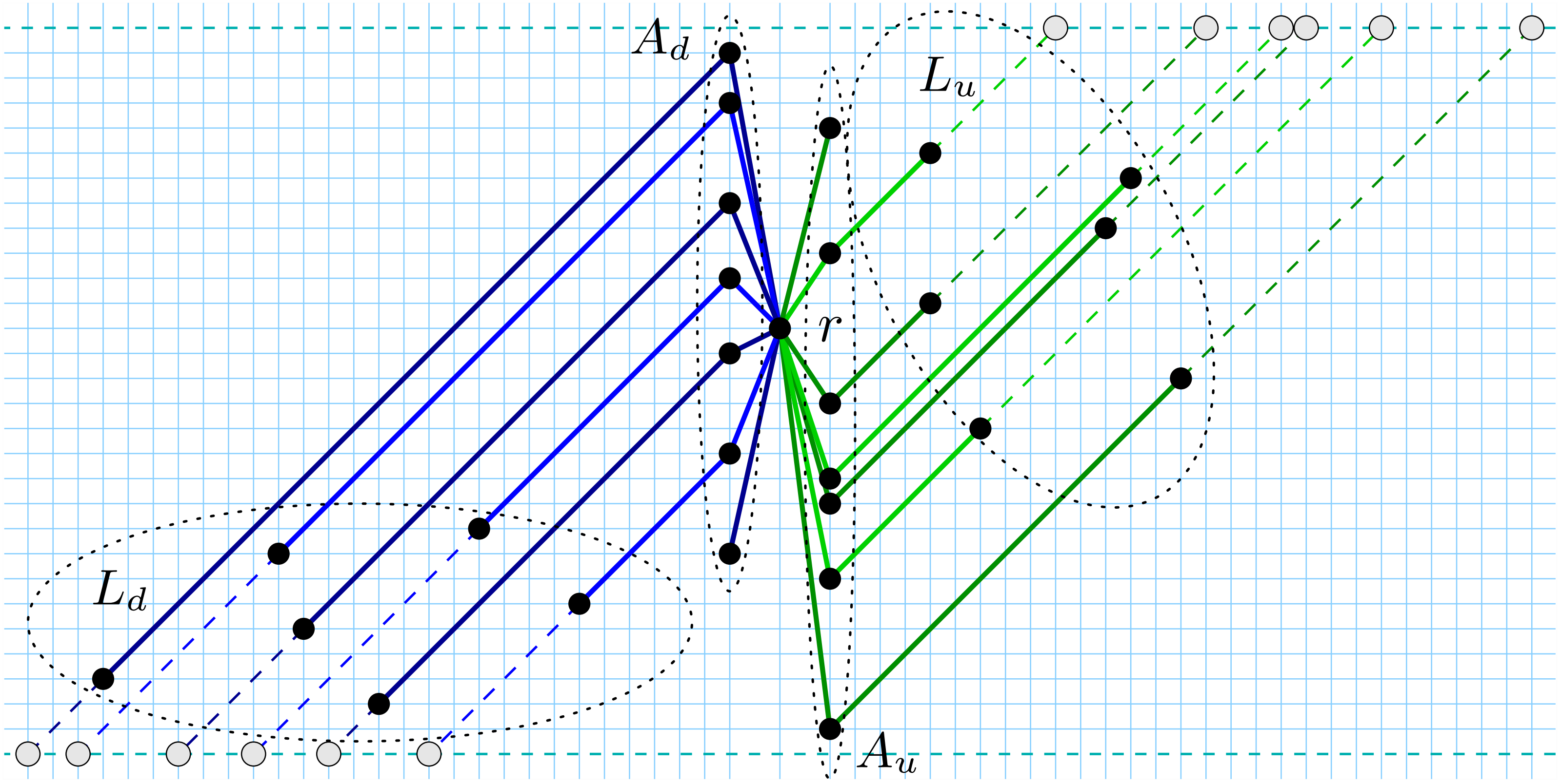

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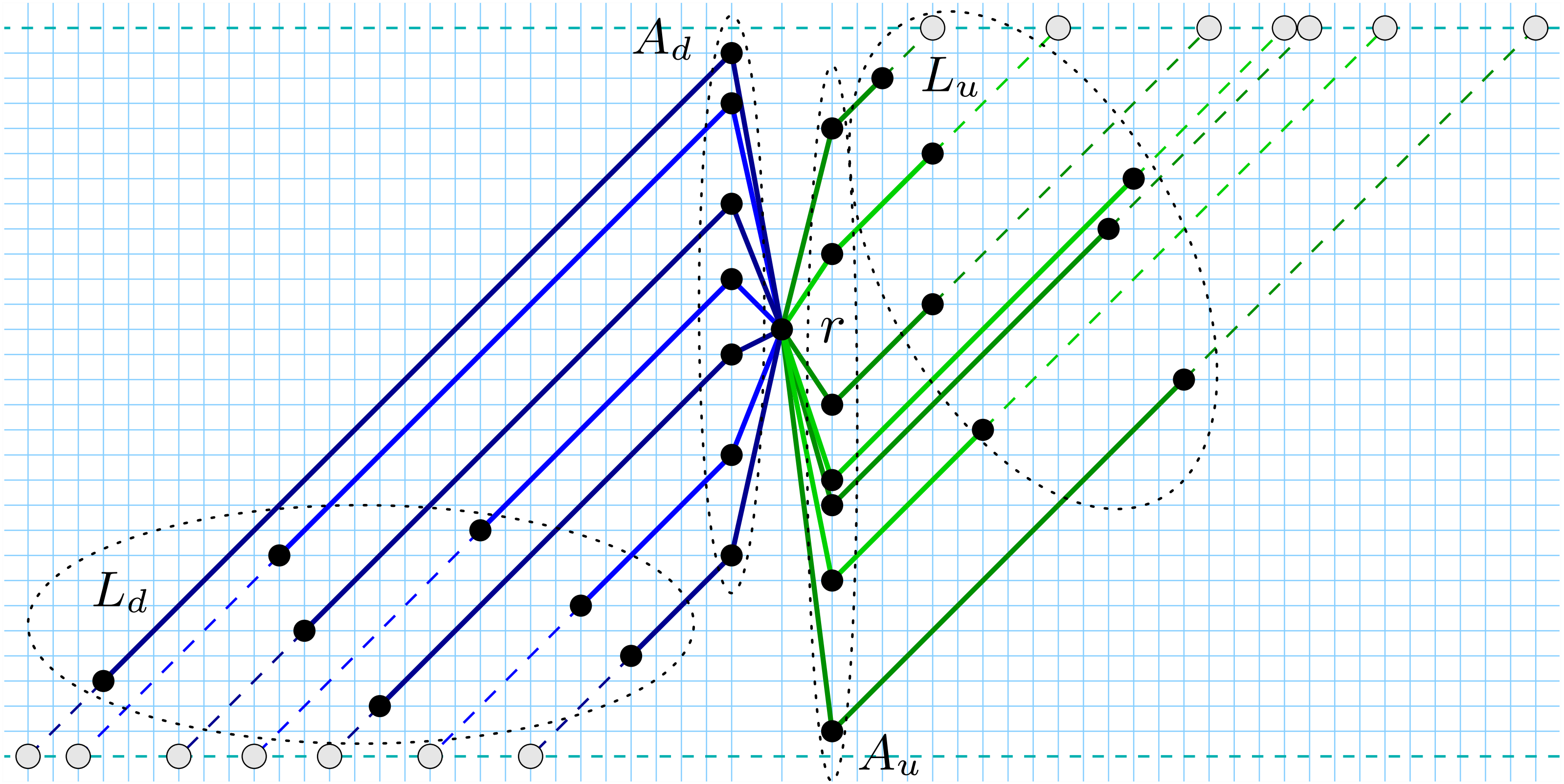

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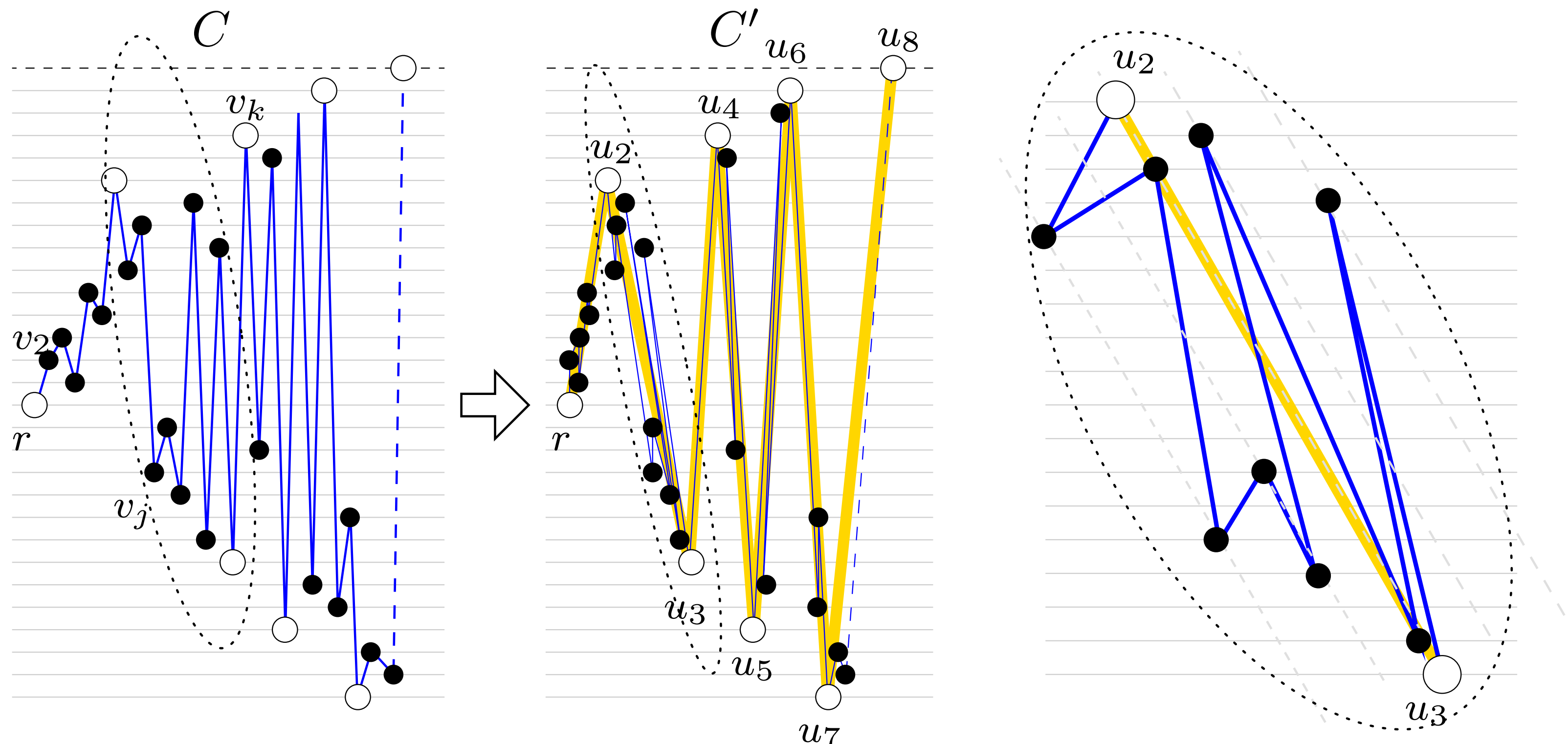

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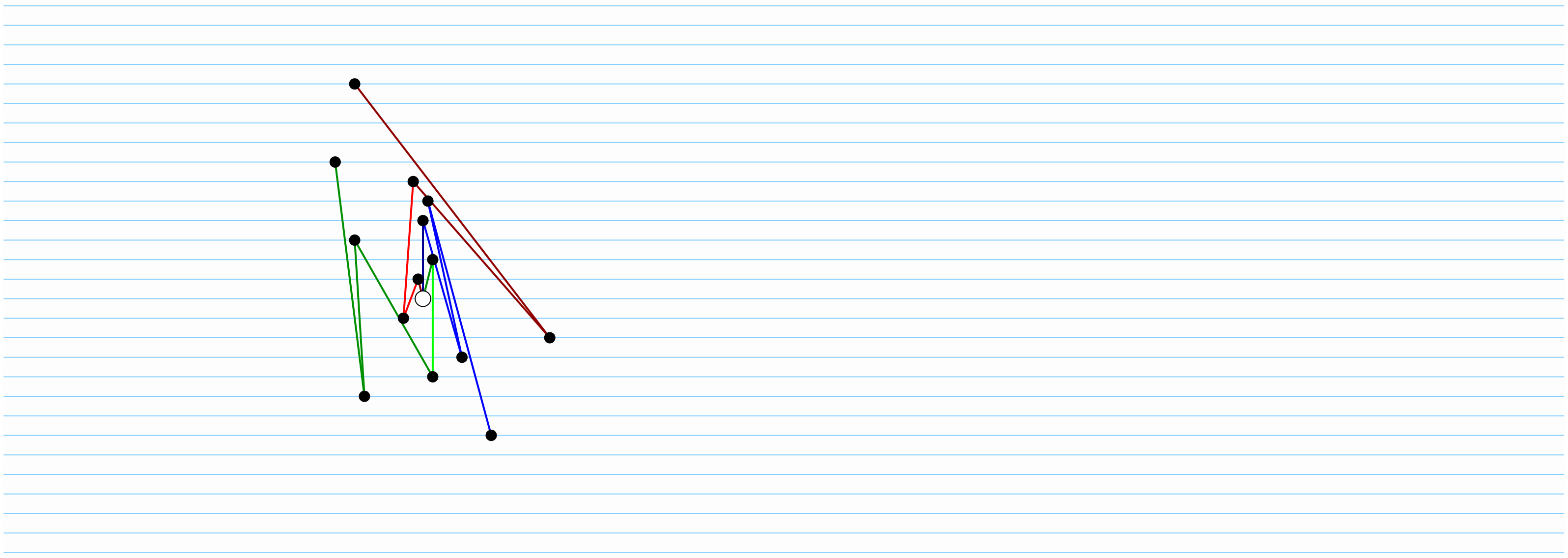

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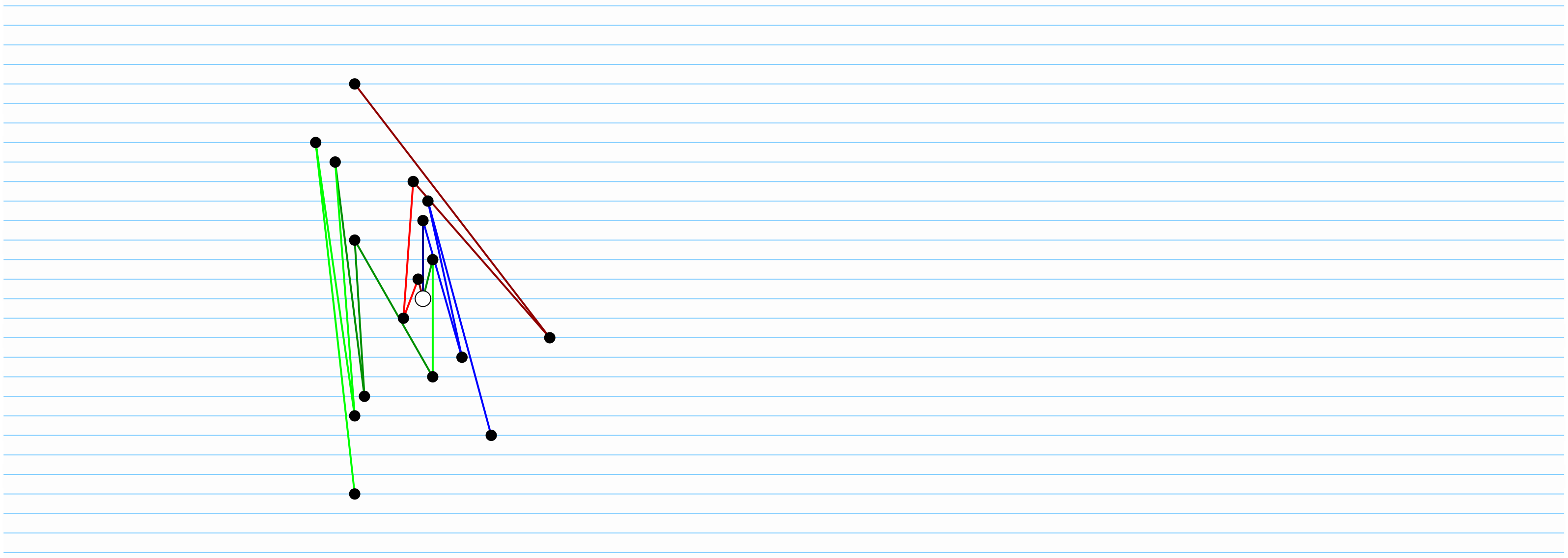

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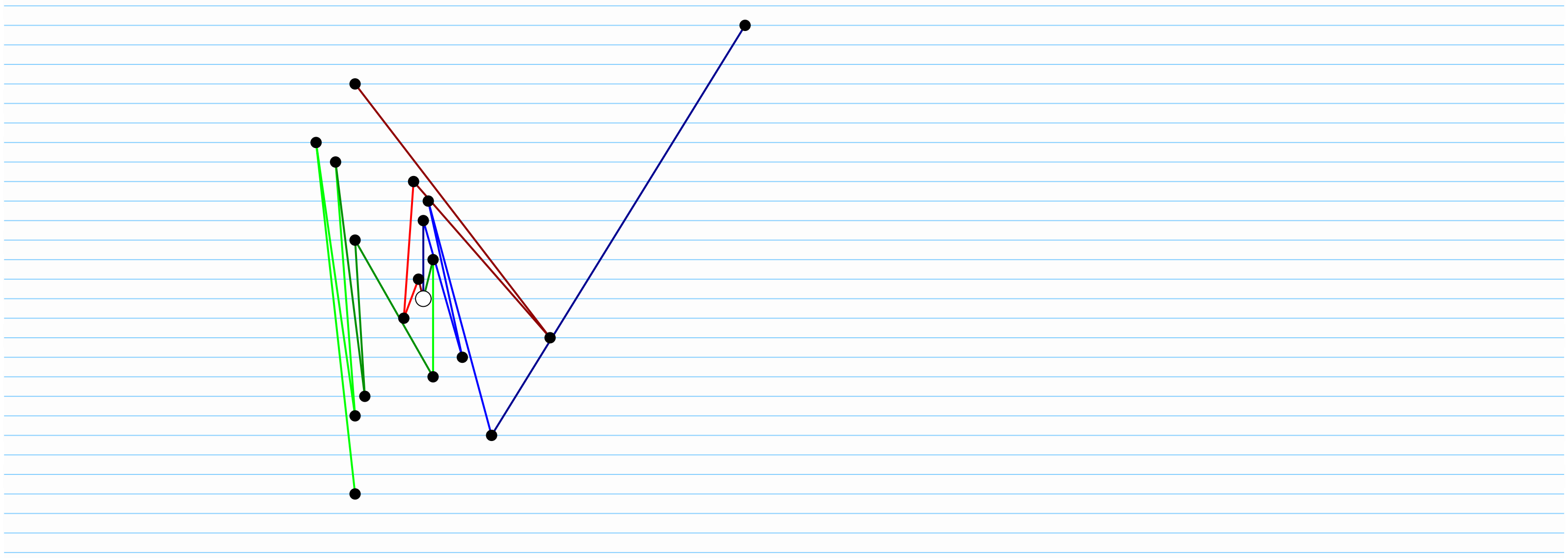

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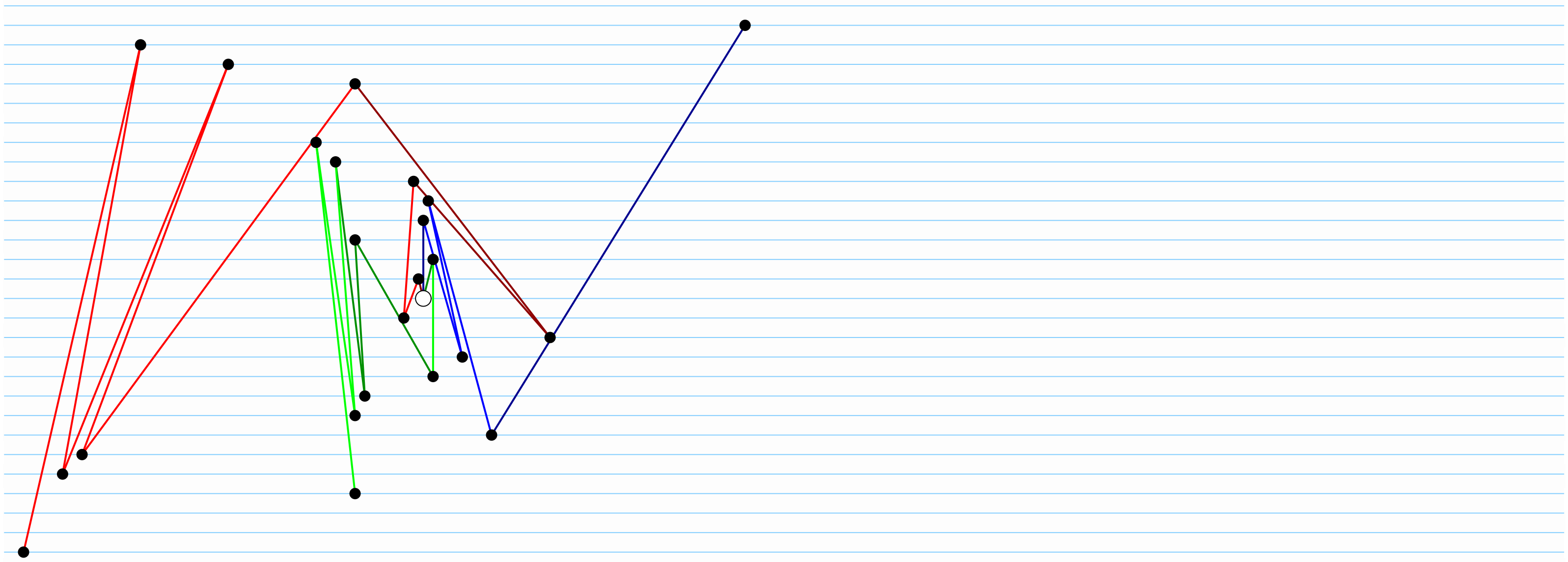

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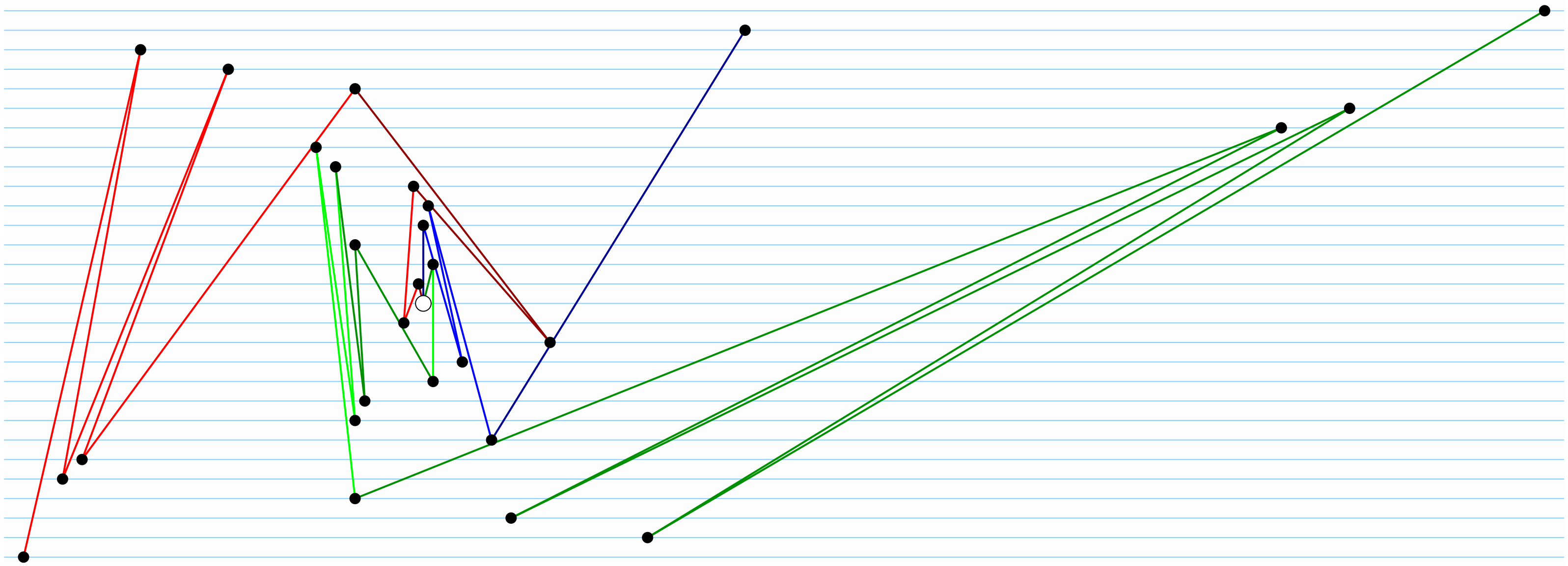

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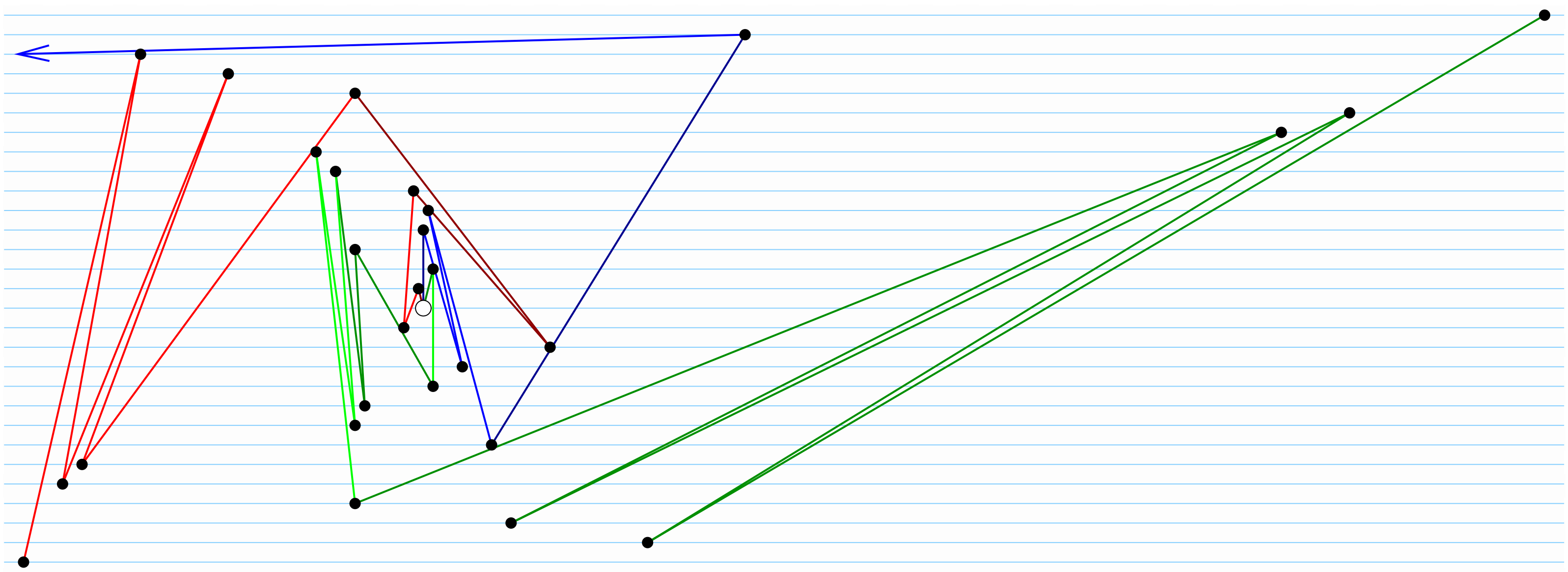

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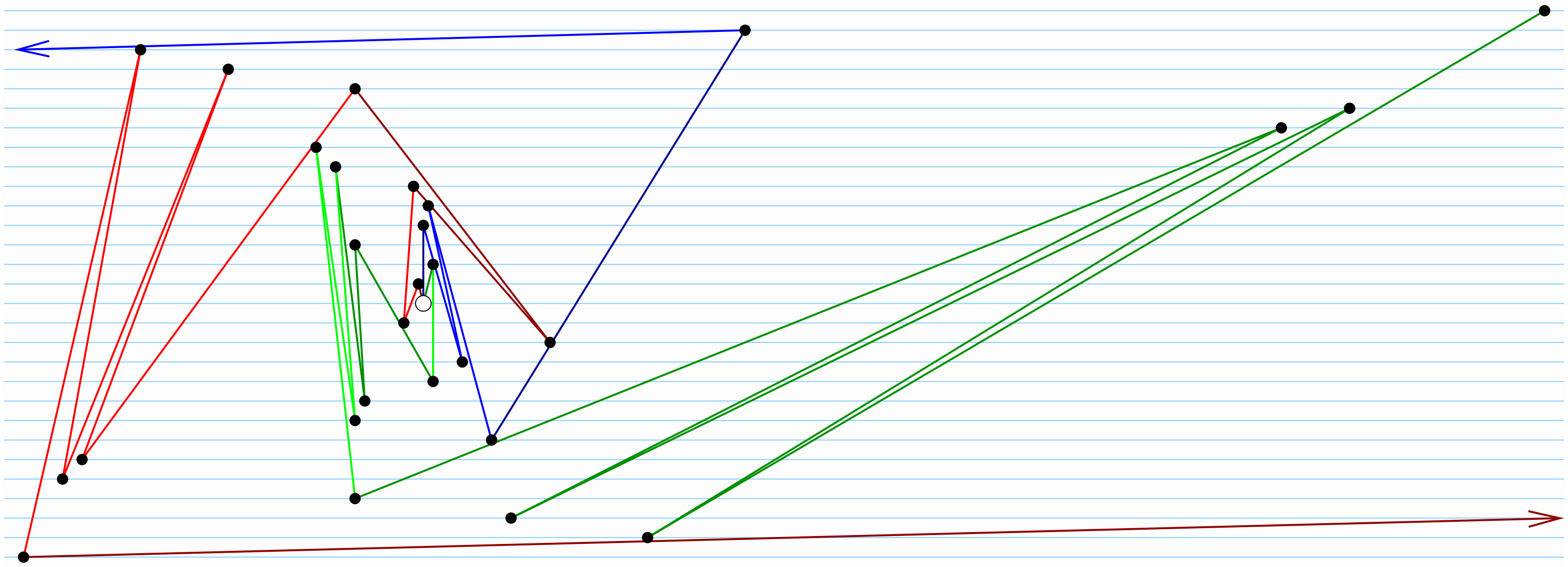
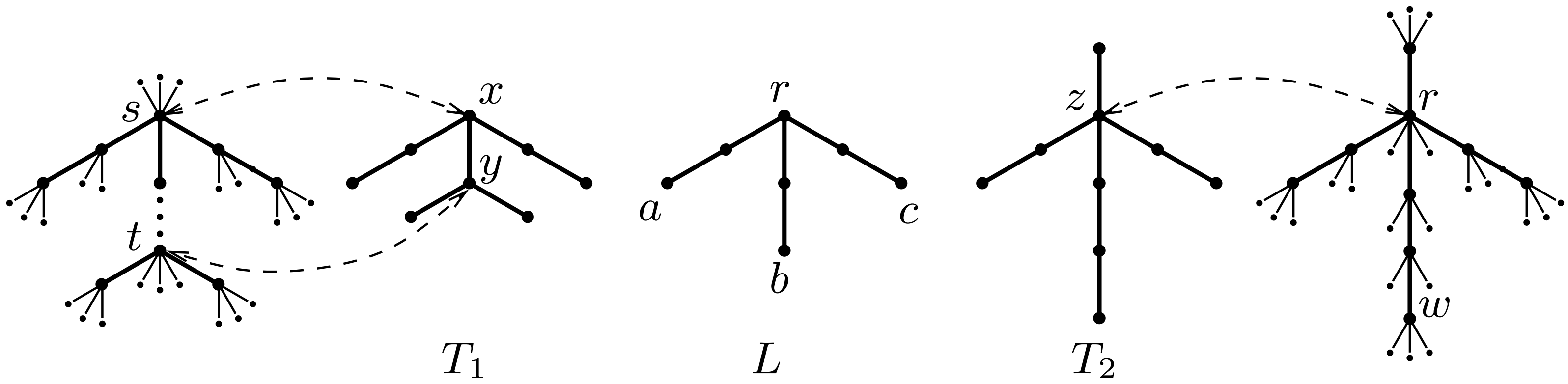


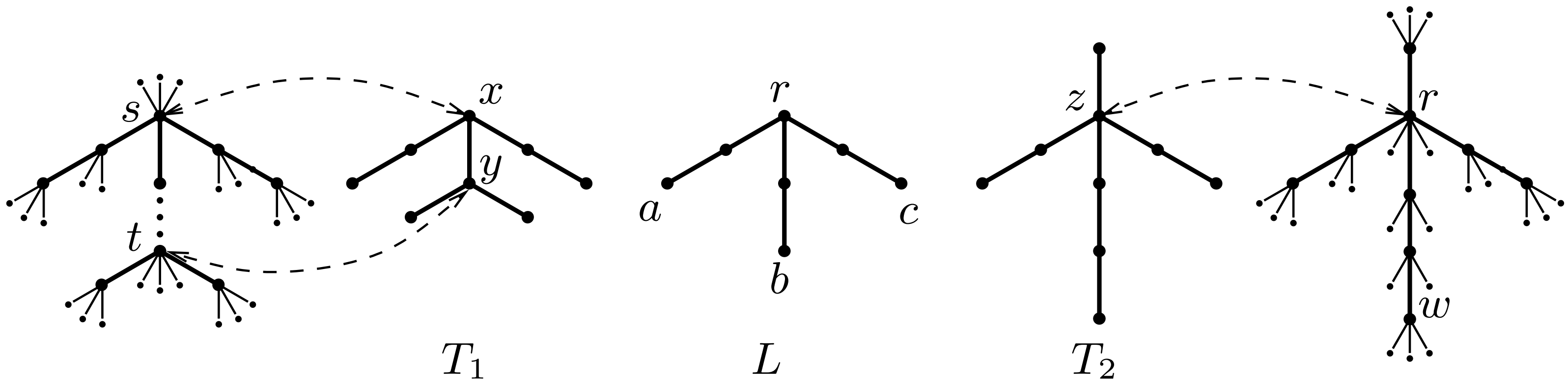
Forbidden Trees – Minimality



- Brute force consideration of the removal of edges from T_1 and T_2 gives the following lemma:

Lemma 6 *Removing any edge from T_1 or T_2 yields a forest of ULP trees.*

Forbidden Trees – Characterization



- A minimal lobster argument gives the next theorem:

Theorem 7 *Every tree either contains a subdivision of T_1 or T_2 in which case it is not ULP, or it is a caterpillar, a radius-2 star, or a 3 spider in which case it is ULP.*

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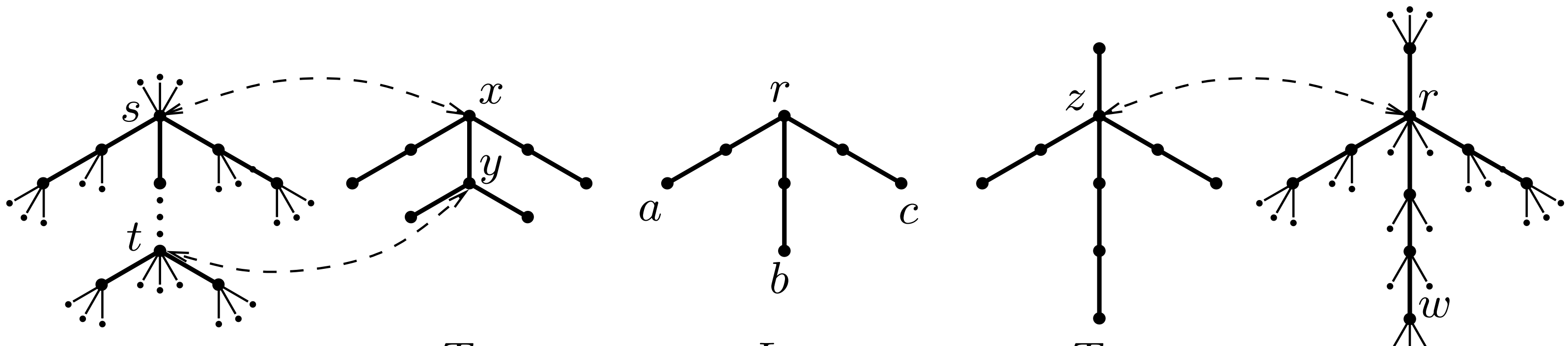

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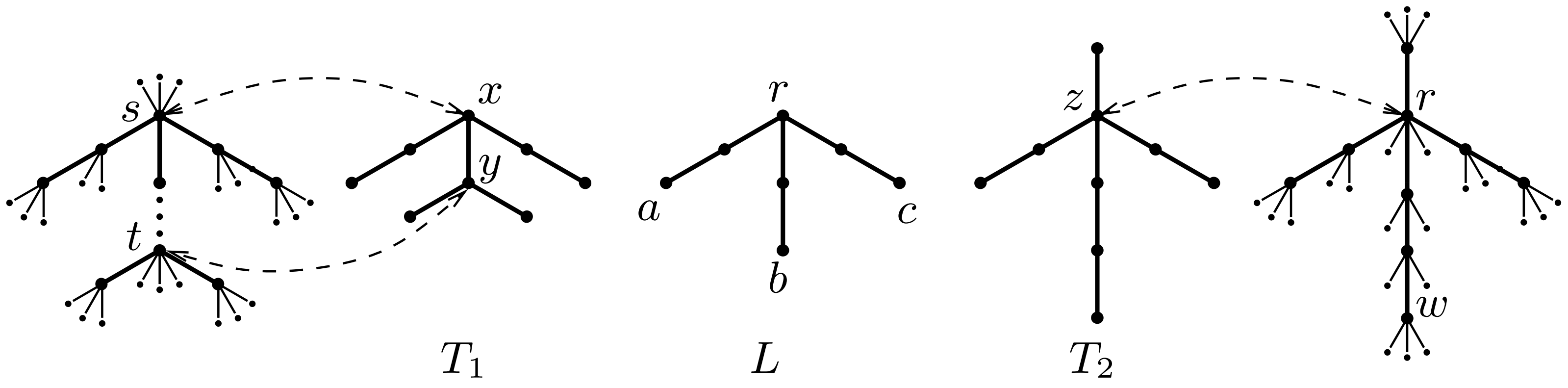

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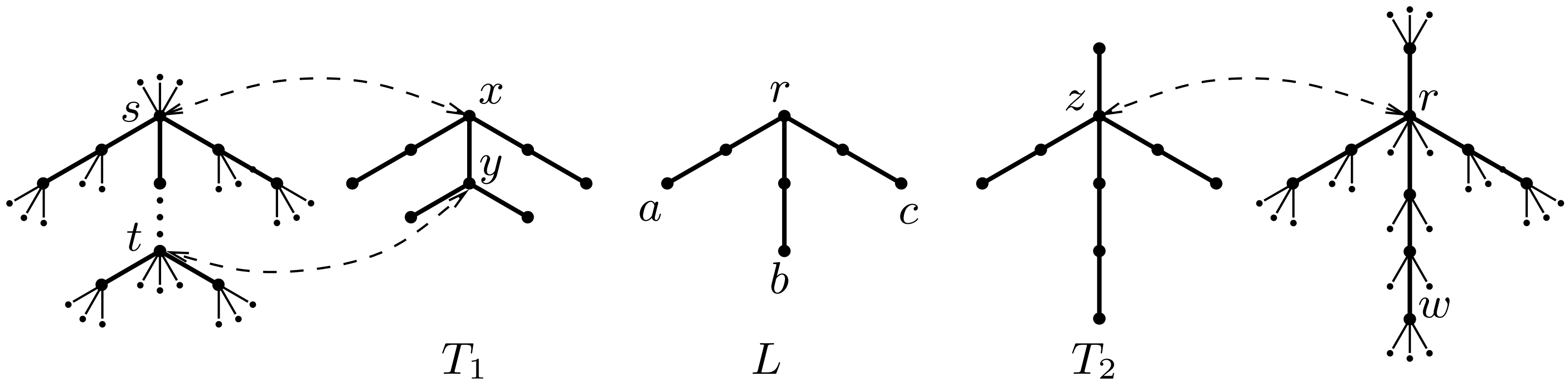

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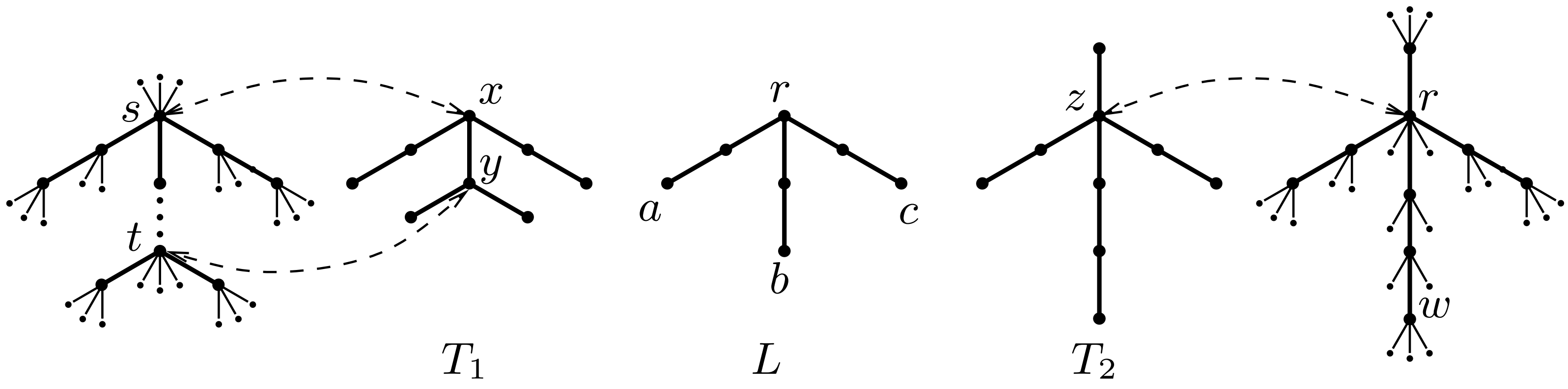

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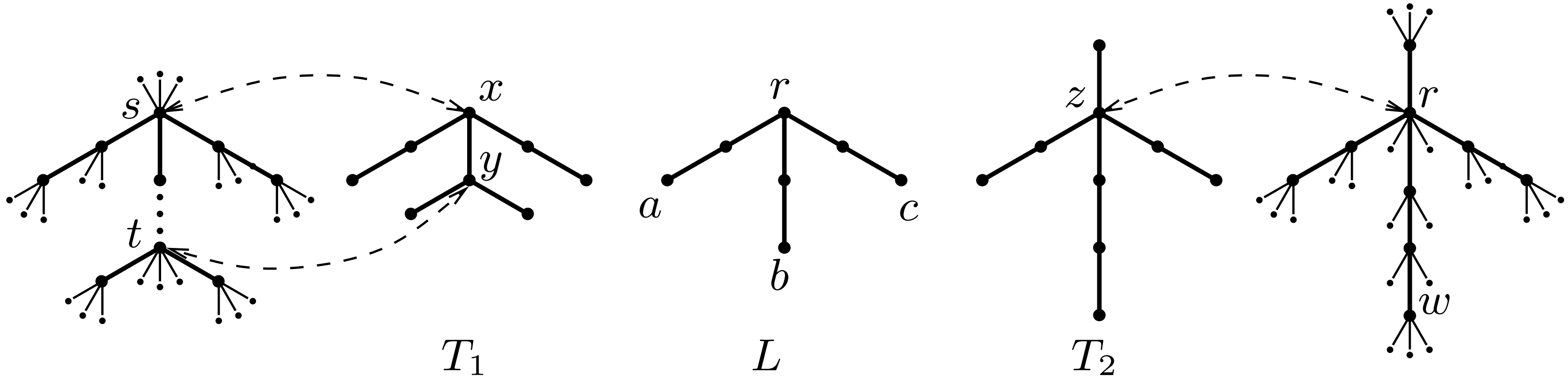

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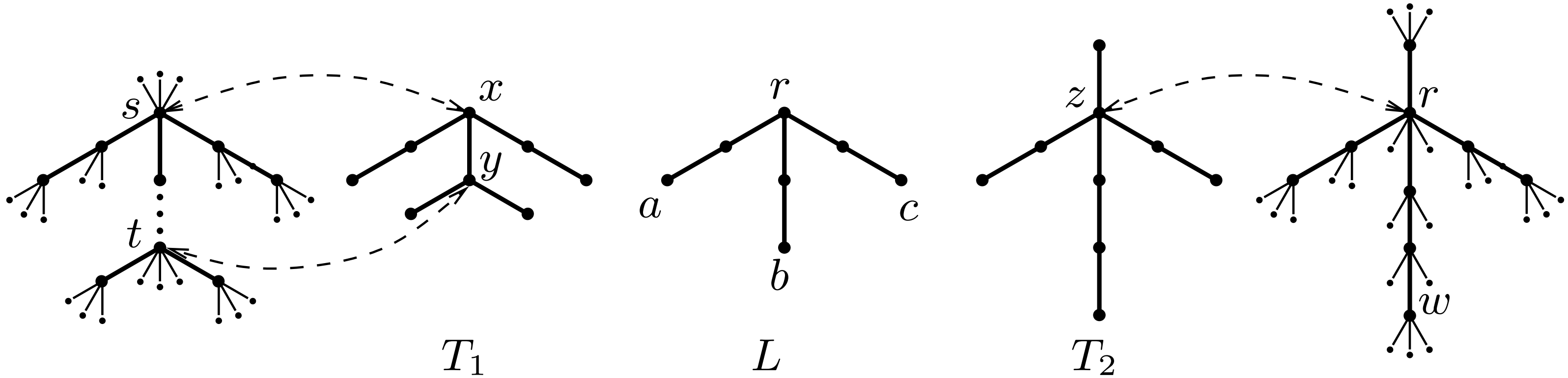

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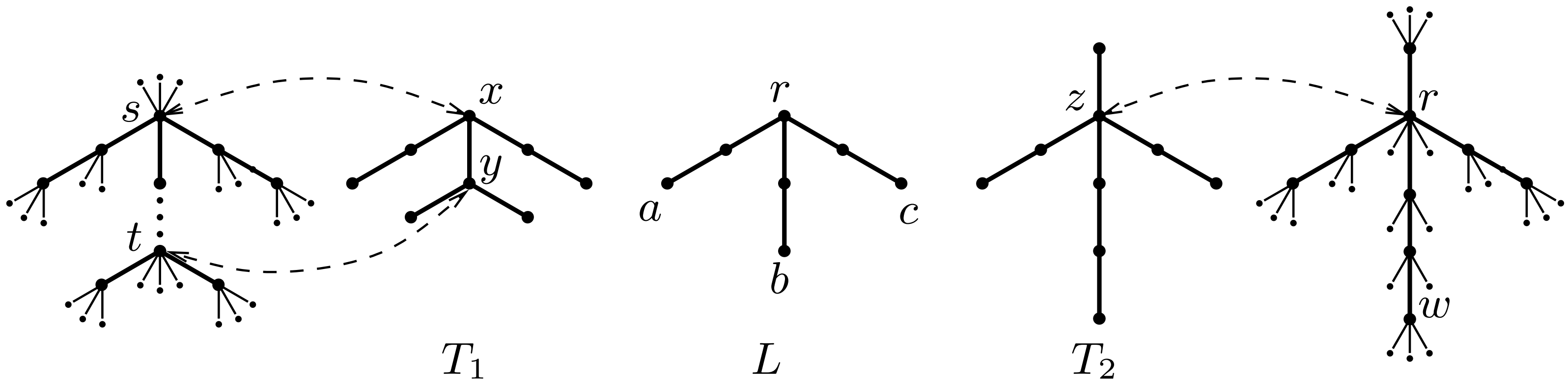
ULP Trees – Recognition



- The final corollary is a consequence of Theorem 7 and properties of degree sequences of caterpillars, radius-2 stars, and degree-3 spiders (Lemmas 8, 9, 10, resp.):

  Corollary 11 *The class of ULP trees can be recognized in linear time.*

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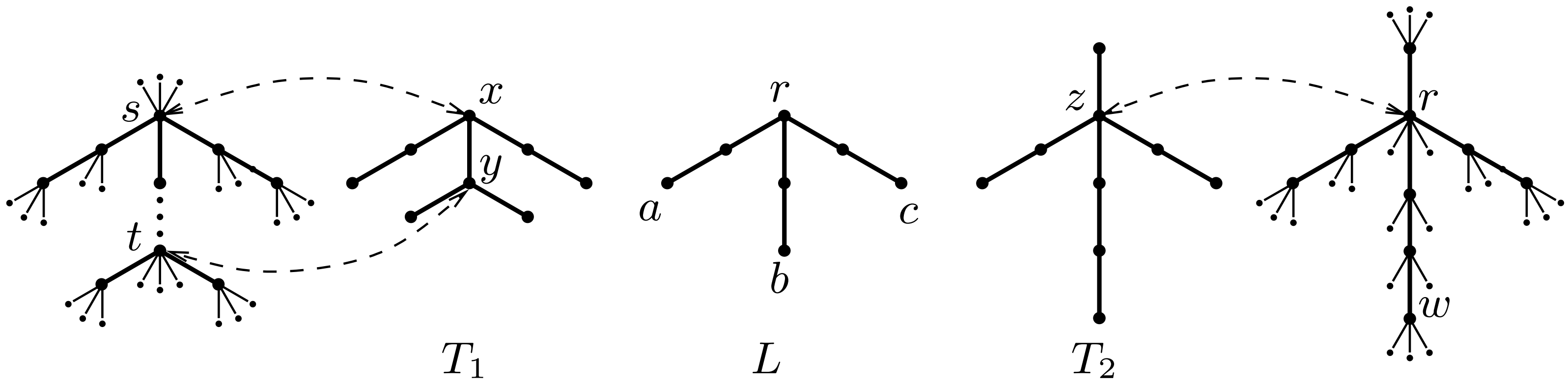

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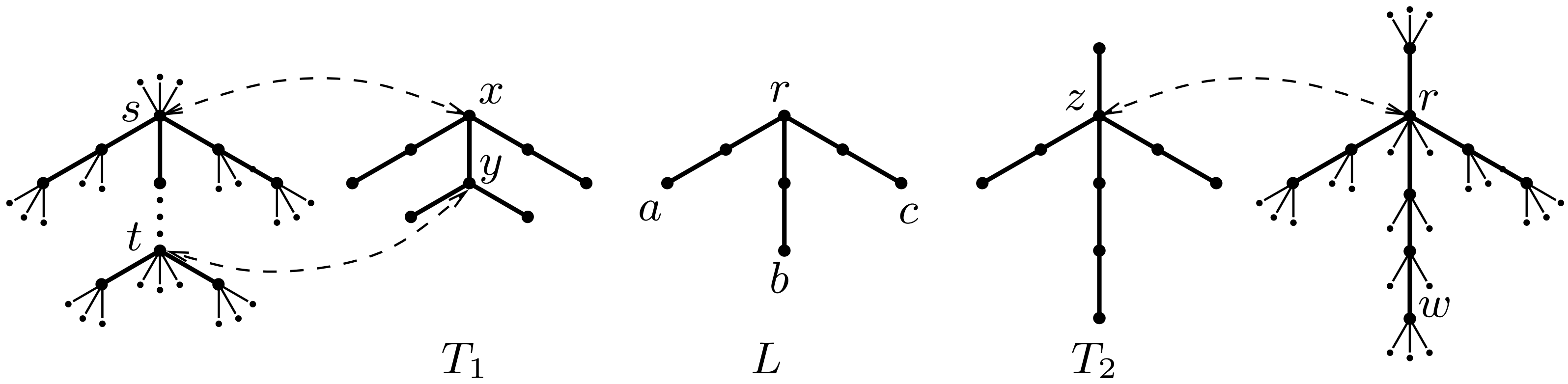

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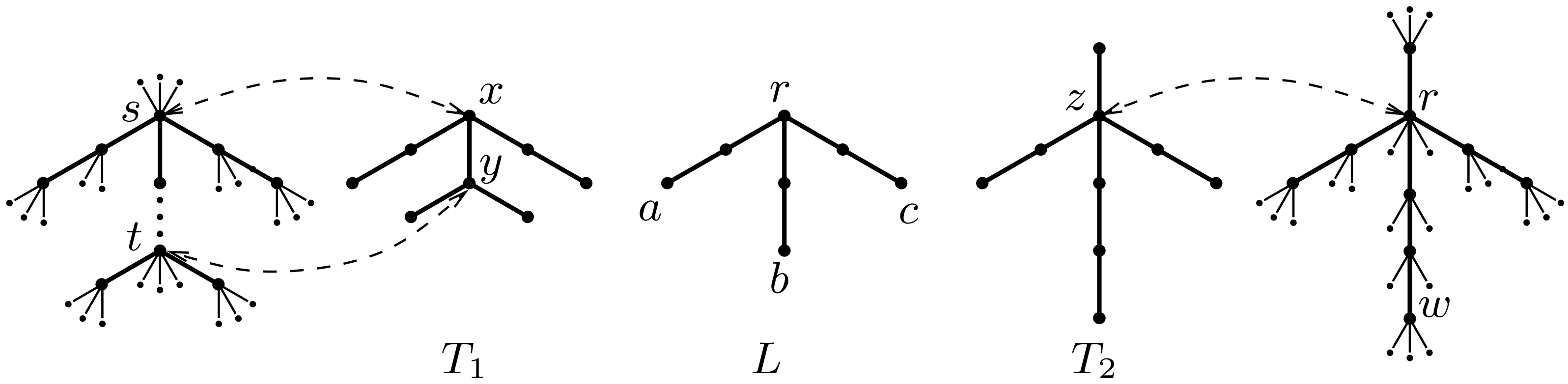

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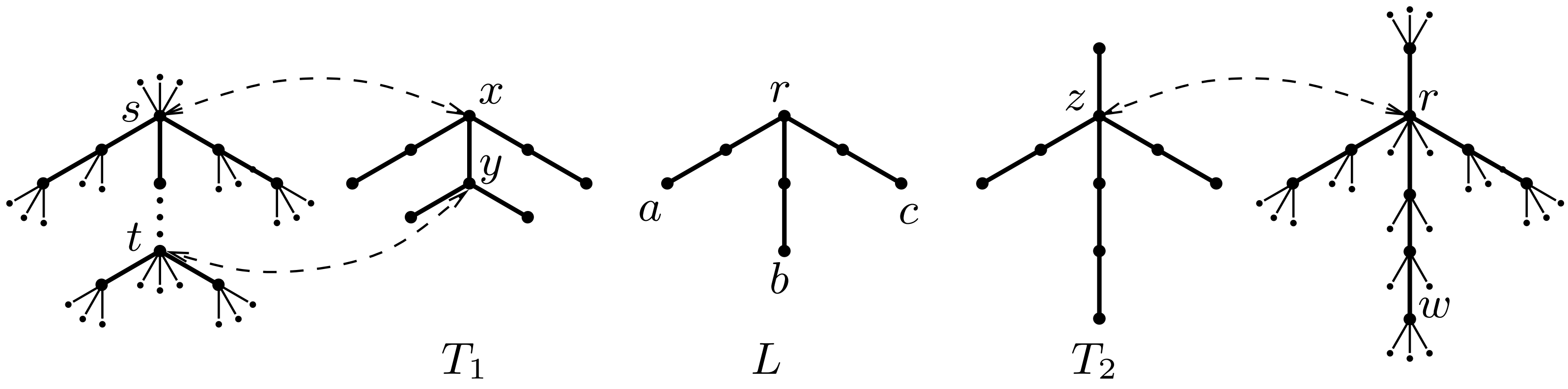

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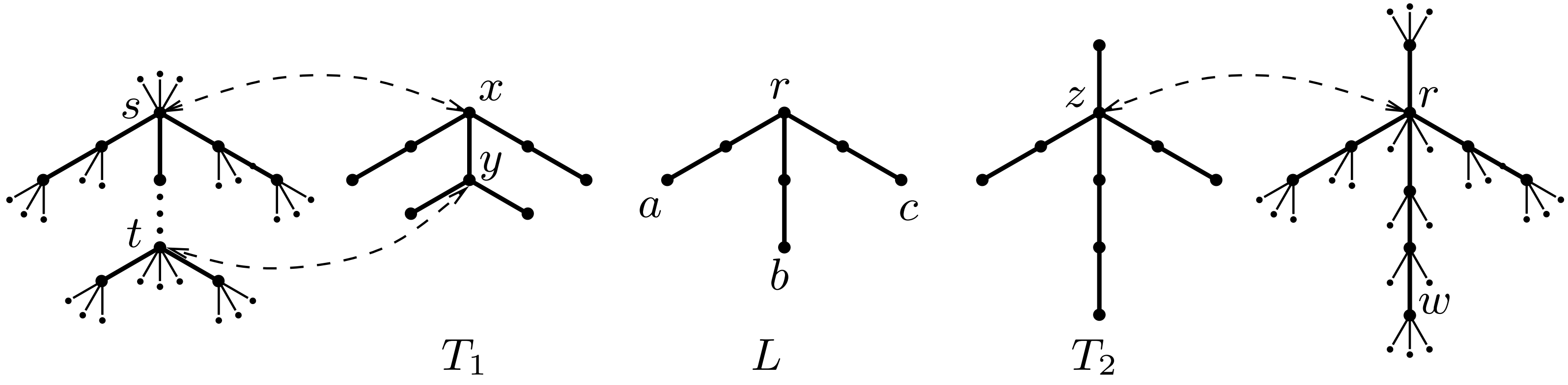

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Future Work

- Provide certificate of unlabeled level non-planarity
  - I.e., find copy of T_1 or T_2

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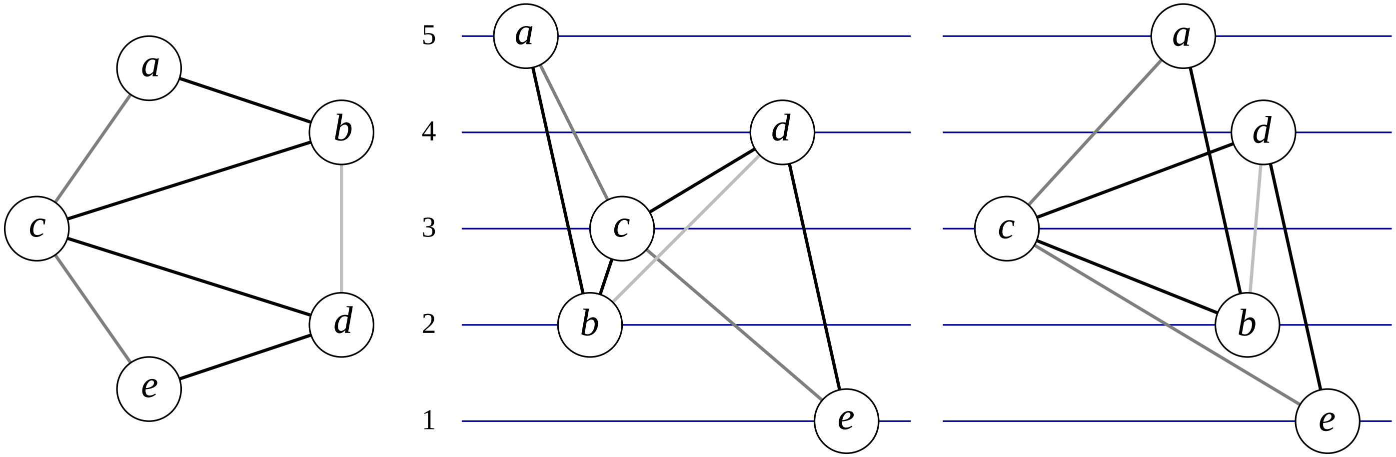
  - First has 5 vertices and two degree 4 vertices



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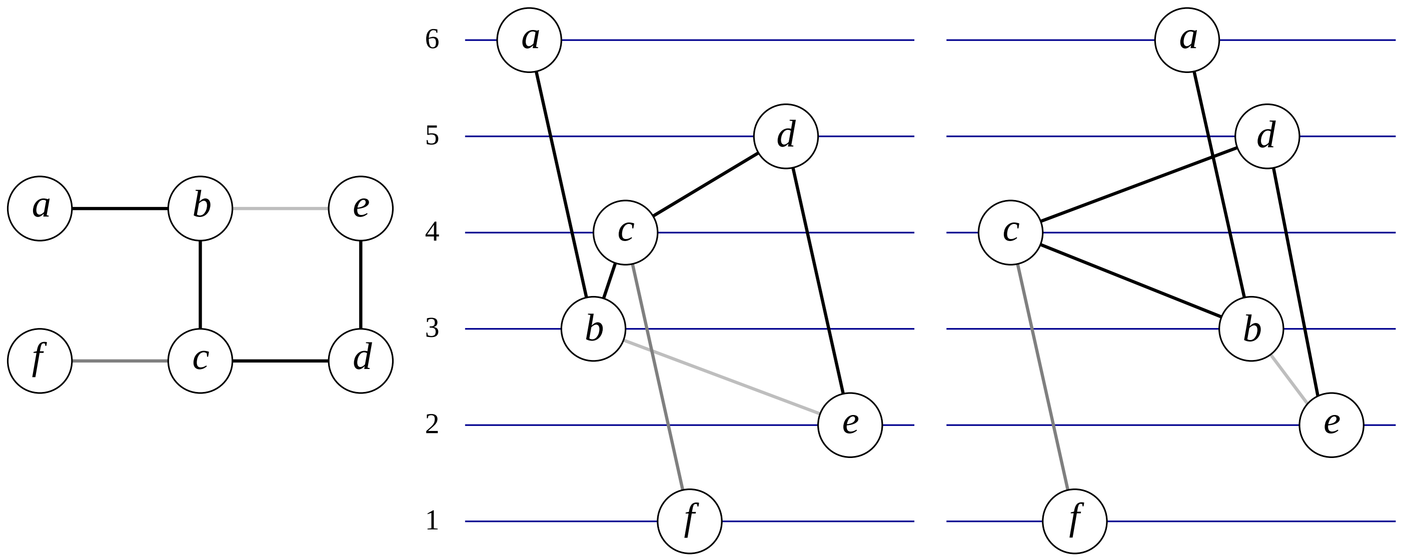
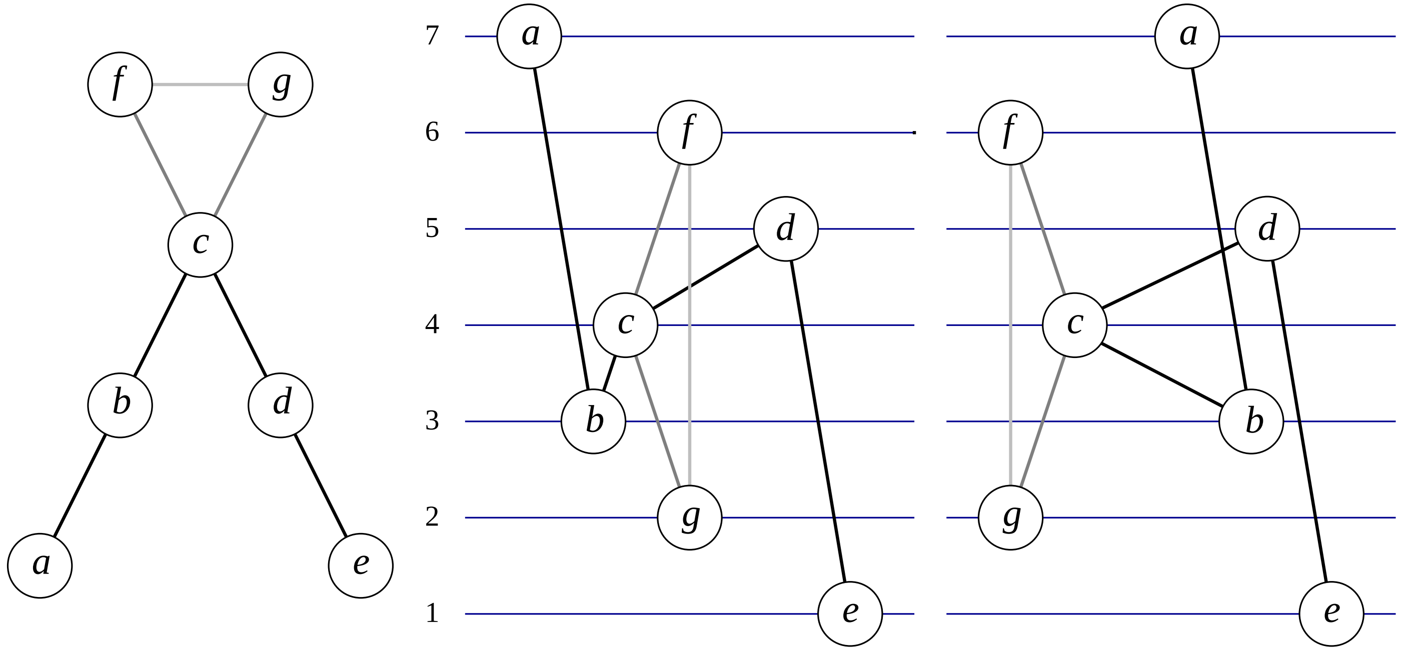


Preview of Future Work

- There are five forbidden ULP subdivisions with cycles
  - Third has 7 vertices and one 3-cycle and one degree 4 vertex



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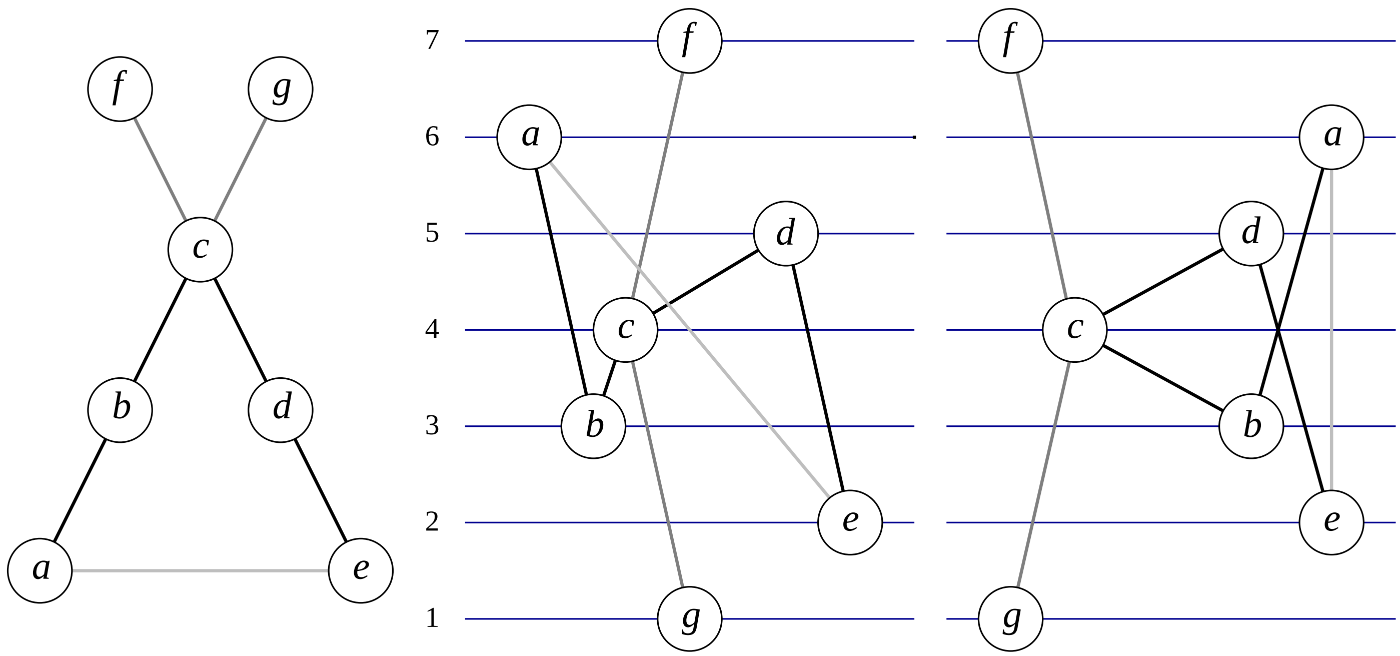

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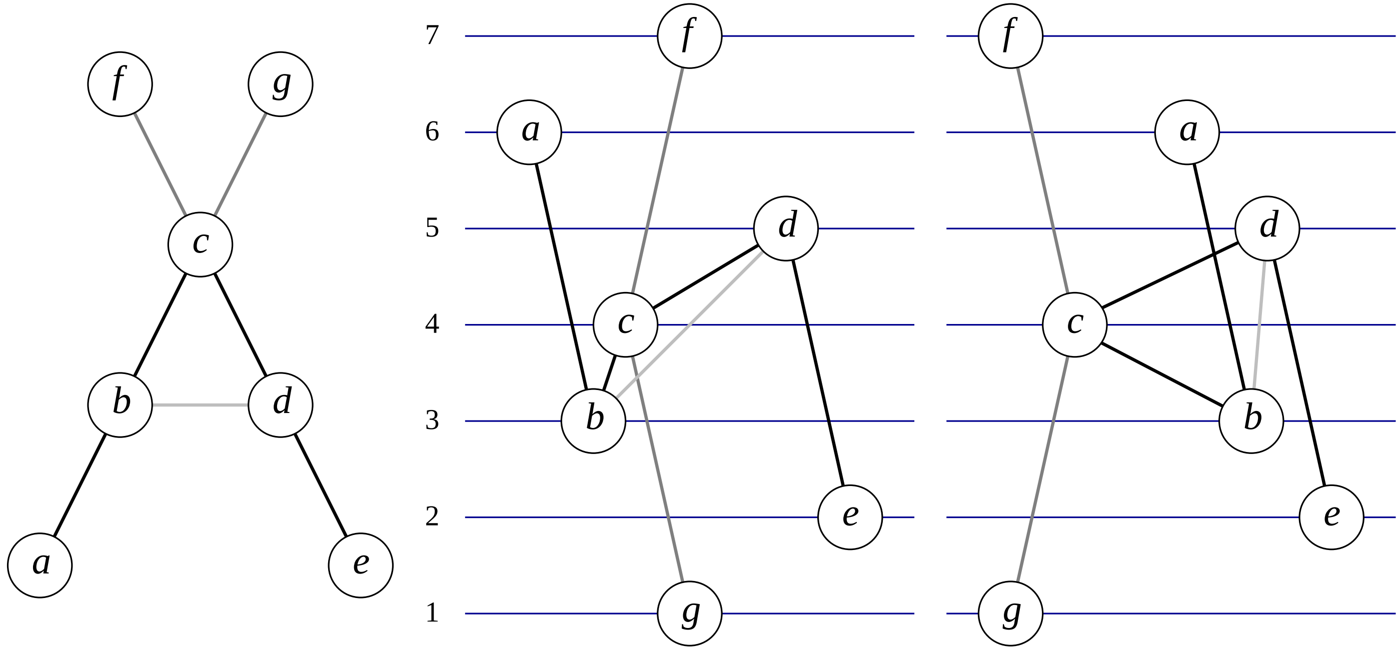

Thank You!