Minimum Level Nonplanar Patterns for Trees

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Abstract. Minimum level nonplanar (MLNP) patterns play the role for level planar graphs that the forbidden Kuratowksi subdivisions K_5 and $K_{3,3}$ play for planar graphs. We add two MLNP patterns for trees to the previous set of tree patterns given by Healy *et al.* [4]. Neither of these patterns match any of the previous patterns. We show that this new set of patterns completely characterizes level planar trees.

1 Introduction

Level graphs model hierarchical relationships. A level drawing has all vertices in the same level with the same y-coordinates and has all edges strictly y-monotone. Level planar graphs have level drawings without edge crossings. Hierarchies are special cases in which every vertex is reachable via a y-monotone path from a source in the top level. Planar graphs are characterized by forbidden subdivisions of K_5 and $K_{3,3}$ by Kuratowksi's Theorem [5]. The counterpart of this characterization for level planar graphs proposed by Healy, Kuusik, and Liepert [4] are the minimum level nonplanar (MLNP) patterns. These are minimal obstructing subgraphs with a set of level assignments that force one or more crossings.

Di Battista and Nardelli [1] provided three level nonplanar patterns for hierarchies (HLNP patterns); cf. Fig. 2. Healy *et al.* adapted these HLNP patterns to MLNP patterns for level graphs. However, the completeness of their characterization was based on the claim that all MLNP patterns must contain a HLNP pattern. We provide a counterexample to this claim based on the level nonplanar assignment for the forbidden tree T_9 used by Estrella *et al.* [2] to characterize the set of unlabeled level planar (ULP) trees; cf. Fig. 1. Healy *et al.* provide two of the MLNP patterns, P_1 and P_2, for trees that are also HLNP patterns; cf. Fig. 3(a) and (b). We provide two more MLNP patterns, P_3 and P_4 for level nonplanar trees; cf. Fig. 3(c) and (d) using our counterexample.

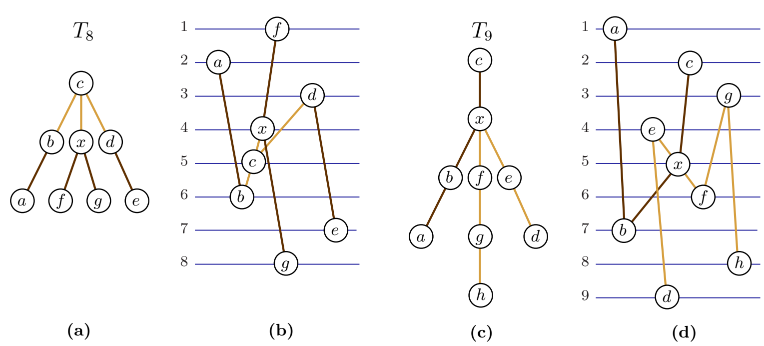

Fig. 1. Labelings preventing the forbidden ULP trees T_8 and T_9 from being level planar.

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2 Preliminaries

A *k-level graph* $G(V, E, \phi)$ on n vertices has *leveling* $\phi : V \to [1..k]$ where every $(u, v) \in E$ either has $\phi(u) < \phi(v)$ if G is directed or $\phi(u) \neq \phi(v)$ if G is undirected. This leveling partitions V into $V_1 \cup V_2 \cup \cdots \cup V_k$ where the *level* $V_j = \phi^{-1}(j)$ and $V_i \cap V_j = \varnothing$ if $i \neq j$. A *proper level graph* only has *short edges* in which $\phi(v) = \phi(u) + 1$ for every $(u, v) \in E$. Edges spanning multiple levels are *long*. A *hierarchy* is a proper level graph in which every vertex $v \in V_j$ for $j > 1$ has at least one incident edge $(u, v) \in E$ to a vertex $u \in V_i$ for some $i < j$.

A *path* p is a non-repeating ordered sequence of vertices $(v_1, v_2, \ldots, v_t)$ for $t \geq 1$. Let $\textsc{min}(p) = \min\{\phi(v) : v \in p\}$, $\textsc{max}(p) = \max\{\phi(v) : v \in p\}$, and $\mathcal{P}(i, j) = \{p : p \text{ is a path where } i \leq \textsc{min}(p) < \textsc{max}(p) \leq j\}$ are the paths between levels V_i and V_j. A *linking path*, or *link*, $L \in \mathcal{L}(i, j)$ is a path $x \rightsquigarrow y$ in which $i = \textsc{min}(L) = \phi(x)$ and $\textsc{max}(L) = \phi(y) = j$, and $\mathcal{L}(i, j) \subseteq \mathcal{P}(i, j)$ are all paths linking the *extreme levels* V_i and V_j. A *bridge* b is a path $x \rightsquigarrow y$ in $\mathcal{P}(i, j)$ connecting links $L_1, L_2 \in \mathcal{L}(i, j)$ in which $b \cap L_1 = x$ and $b \cap L_2 = y$.

A *level drawing* of G has all of its *level-j* vertices in the j^{th} level V_j placed along the *track* $\ell_j = \{(x, k - j) \,|\, x \in \mathbb{R}\}$, and each edge $(u, v) \in E$ is drawn as a continuous strictly y-monotone sequence of line segments. A level drawing drawn without edge crossings shows that G is *level planar*. A *pattern* is a set of level nonplanar graphs sharing structural similarities. Removing any edge from the underlying graph matching a *minimum level nonplanar* (MLNP) *pattern* gives a level planar graph. A *hierarchy level nonplanar* (HLNP) *pattern* is a level nonplanar pattern in which every matching graph is a hierarchy. The next theorem gives the set of the three distinct HLNP patterns.

Theorem 1 *[Di Battista and Nardelli [1]] A hierarchy $G(V, E, \phi)$ on k levels is level planar if and only if there does not exist three paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \leq i < j \leq k$ where one of the following hold:*

- (P_A) *L_1, L_2, and L_3 are completely disjoint and pairwise connected by bridges b_1, b_2, b_3 where $b_1 \cap L_2 = b_2 \cap L_1 = b_3 \cap L_1 = \varnothing$; cf. Fig. 2(a).*
- (P_B) *L_1 and L_2 share a path $C = L_1 \cap L_2$ from $p \in V_i \cup V_j$ where $L_1 \cap L_3 = L_2 \cap L_3 = \varnothing$ are connected by bridges b_1 from L_1 to L_3 and b_2 from L_1 to L_3 such that $b_1 \cap L_2 = b_2 \cap L_1 = \varnothing$; cf. Fig. 2(b).*
- (P_C) *L_1 and L_2 share a path $C_1 = L_1 \cap L_2$ from $p \in V_i$ and L_2 and L_3 share a path $C_2 = L_2 \cap L_3$ from $q \in V_j$ such that $C_1 \cap C_2 = \varnothing$. Bridge b connects L_1 and L_3 where $b \cap L_2 = b \cap C_1 = b \cap C_2 = \varnothing$; cf. Fig. 2(c).*

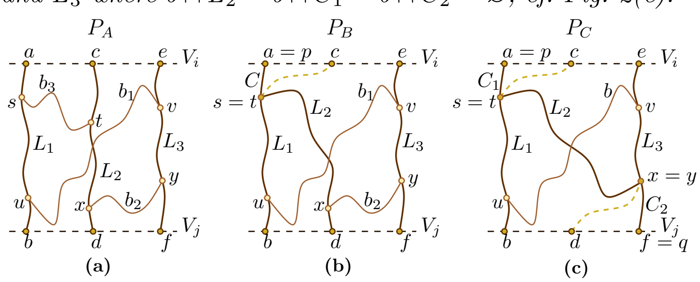

Fig. 2. The three patterns characterizing hierarchies.

3 MLNP Patterns for Trees

Theorem 2 *A level tree $T(V, E, \phi)$ on k levels is minimum level nonplanar if*

(1) there are three disjoint paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ for $1 \le i < j \le k$ where P_A of Theorem 1 applies and the union of the three bridges $b_1 \cup b_2 \cup b_3$ forms a subdivided $K_{1,3}$ subtree S with vertex c of degree 3 where either

(P_1) $c \in V_i$ (or V_j) and there is a leaf of S in V_j (or V_i) as in Fig. 3(a) or

(P_2) one leaf of S is in V_i and another leaf of S is in V_j as in Fig. 3(b), or

(2) there are four paths $L_1, L_2, L_3, L_4 \in \mathcal{L}(i, j)$ for $1 \le i < j \le k$ where $L_1 \cap L_4 = \varnothing$, $L_1 \cap L_2 \in V_j$ (or V_i) and $L_3 \cap L_4 \in V_i$ (or V_j) where $L_1 \cup L_2$ and $L_3 \cup L_4$ form paths with both endpoints in V_i and V_j (or V_j and V_i), resp., and there exist levels V_l and V_m for some $i < l < m < j$ in which either L_2 or L_3 consists of subpaths $C_1 \in \mathcal{L}(i, m)$, $C_2 \in \mathcal{L}(l, m)$, and $C_3 \in \mathcal{L}(l, j)$ where either

(P_3) $L_2 \cap L_3 = x$ where $l \le \phi(x) \le m$ as in Fig. 3(c), or

(P_4) $L_2 \cap L_3$ is path $x \rightsquigarrow y$ where $l \le \{\phi(x), \phi(y)\} \le m$ and $L_2 = c \rightsquigarrow x \rightsquigarrow y \rightsquigarrow b$ where $c \in V_i$ (or V_j) and $b \in V_j$ (or V_i) as in Fig. 3(d).

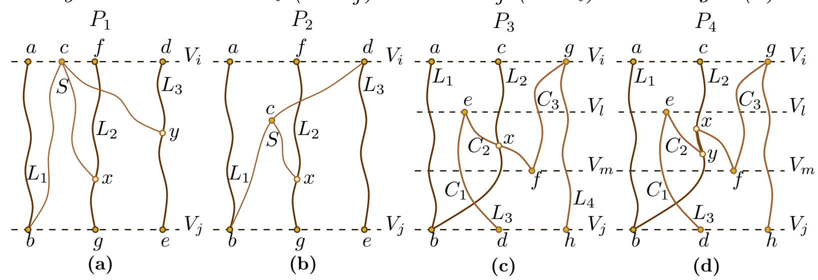

Fig. 3. Four MLNP patterns for trees.

Proof. P_1 and P_2 are MLNP given they match T1 and T2 of Healy *et al.* The argument in [2] used by Estrella *et al.* to show T_9 is level nonplanar generalizes for P_3 and P_4. To see that P_3 is minimal (P_4 is similar), we try the seven distinct ways of removing an edge; cf. Fig. 4. In each case crossings can be avoided. □

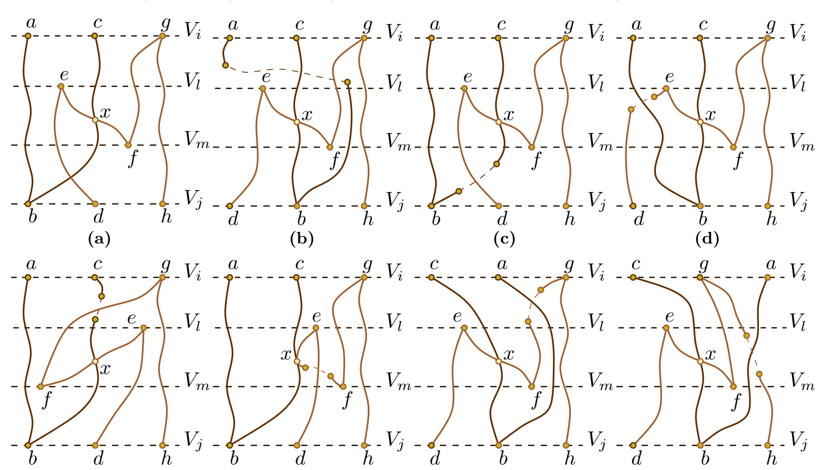

Fig. 4. The seven cases of deleting an edge from pattern P_3 in (a).

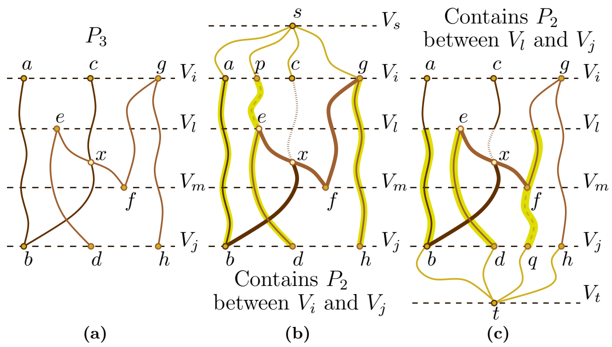

Fig. 5. Augmenting P_3 in (a) from above (b) and below (c) to form hierarchies.

The proof of Theorem 15 of Healy *et al.* [4] argues that every MLNP pattern must match some HLNP pattern. We show why this argument fails for P_3.

Lemma 3 *P_3 augmented to form a hierarchy has a subtree matching P_2.*

Proof. Fig. 5 shows the highlighted subtrees that match P_2 when P_3 is augmented to form a hierarchy. However, P_2 does not match P_3 by Theorem 2. $\square$

The next lemma gives the minimal conditions for a MLNP tree pattern.

Lemma 4 *A level nonplanar tree $T(V, E, \phi)$ on k levels contains three disjoint paths $L_1, L_2, L_3 \in \mathcal{L}(i, j)$ linking levels V_i and V_j for $1 \le i < j \le k$ with bridges b_1 from L_1 to L_2 and b_2 from L_2 to L_3 with $x = b_1 \cap L_2$ and $y = b_2 \cap L_2$ so that either (P_α) $x = y$, (P_β) $L_2 = c \rightsquigarrow y \rightsquigarrow x \rightsquigarrow d$, or ($P_\gamma$) $L_2 = c \rightsquigarrow x \rightsquigarrow y \rightsquigarrow d$ hold where $c \in V_i$ and $d \in V_j$ as in Fig. 6(a), (b), (c).*

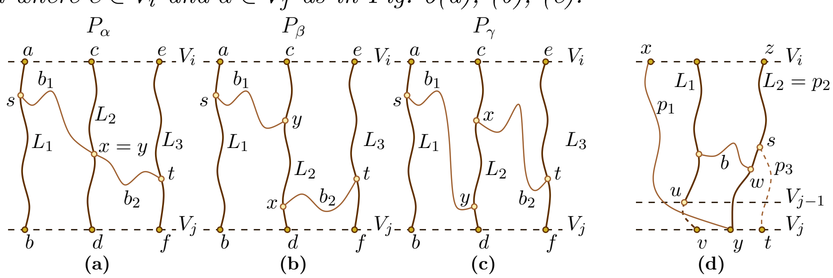

Fig. 6. Minimal patterns for Lemma 4.

Proof. Assume that P is an MLNP pattern between levels V_i and V_j in which $|i - j|$ is minimum and there are at most two disjoint paths $L_1, L_2 \in \mathcal{L}(i, j)$. There could be at most one bridge b joining L_1 and L_2 without forming a cycle. Let w be the endpoint of b in L_2. Let P' be $P - (u, v)$ where (u, v) is the short edge connecting L_1 to V_j in which $v \in V_j$. In order for P to be MLNP, there must exist two linking paths $p_1, p_2 \in \mathcal{L}(i, j)$ in P' with endpoints $x, z \in V_i$ and common endpoint $y \in V_j$ such that for any level planar embedding of P', u is contained in the region bounded by p_1, p_2 and the track ℓ_i; cf. Fig. 6(d). Assume w.l.o.g. that L_2 is p_2. In order for p_1 not to be embeddable on the other side of p_2 (allowing edge (u, v) to be drawn in P without crossing), there must be a path p_3 from s in L_2 to $t \in V_j$ in which s lies between z and w blocking this direction. Then there are at least three disjoint paths in P in $\mathcal{L}(i, j)$: p_1, L_1 and the path $z \rightsquigarrow s \rightsquigarrow t$, contradicting our assumption of there only being two.

Let $L_1, L_2, L_3 \in \mathcal{L}(i,j)$ be three disjoint paths. At least one of the three paths, say it is L_2, must be joined by bridges b_1 and b_2 to the other two paths L_1 or L_3, respectively, or P would be disconnected contradicting the minimality of P. If $b_1 \cap b_2$ form a nonempty path, then $b_1 \cup b_2$ would form a subtree homeomorphic to $K_{1,3}$, yielding pattern P_1 or P_2 of Theorem 2. Thus, b_1 and b_2 can share at most one vertex as in P_α of Fig. 6(a). Otherwise there must have been endpoints $x = b_1 \cup L_2$ and $y = b_2 \cup L_2$ along the path $c \rightsquigarrow d$ forming L_2 where either y proceeds x as in P_β of Fig. 6(b) or x proceeds y as in P_γ of Fig. 6(c). □

We next show that P_4 is easily derived from P_3.

Lemma 5 *P_4 is the only distinct* MLNP *pattern for trees that can be formed from P_3 (by splitting the degree-4 vertex) not containing a subtree matching P_2.*

Proof. Fig. 7 shows the three ways the degree-4 vertex of P_3 can be split into two degree-3 vertices. Two contain subtrees that match P_2. □

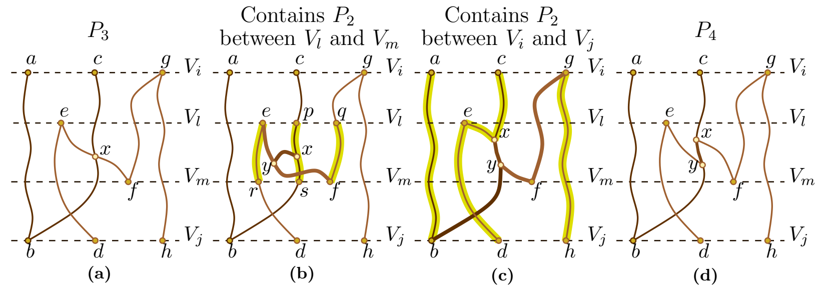

Fig. 7. The three ways of splitting the degree-4 vertex of P_3 into two vertices of degree 3

Finally we complete our characterization for level nonplanar trees.

Theorem 6 *A level tree T is level nonplanar if and only if T has a subtree matching one of the minimum level nonplanar patterns P_1, P_2, P_3, or P_4.*

Proof Sketch: We sketch proof for the simplest case here; the full proof can be found in [3]. Once a MLNP pattern P is augmented to form a hierarchy, one of the HLNP patterns must apply. Since this augmentation does not introduce a cycle between levels V_i and V_j, either pattern P_1 or P_2 must match a subtree of the augmented pattern by Lemma 5 of [4].

Assume there is a MLNP tree pattern P containing P_α of Lemma 4 that does not match P_1 or P_2. We consider the simplest case of how the bridges of P_α in P could spans levels between V_i and V_j. We augment P to form a hierarchy to illustrate how either P must match P_1 or P_2 or contain a cycle.

Suppose that a bridge of P_α in P is not strictly y-monotone. Then P could either have a bend at e in level V_l in one bridge or a bend at f in level V_m in the other as in Fig. 8(a) for some $i < l < m < j$. Each bend would require augmentation to a path from the source when forming a hierarchy from above or below as was the case with P_3 in Fig. 5.

We augment P with a path $p \rightsquigarrow e$ from V_i to V_l to form P', a hierarchy, that must match P_1 or P_2. We observe that between levels V_i and V_m, we have four linking paths. A third bridge $u \rightsquigarrow v$ must be present in P' that is part of a subtree S homeomorphic to $K_{1,3}$. Fig. 8(b) gives one such example. While P'

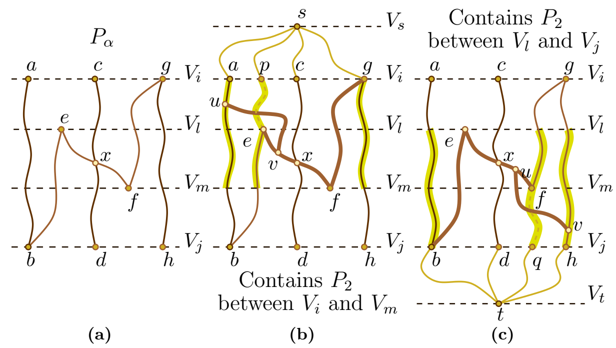

Fig. 8. Examples of pattern P_α in (a) being augmented to form a hierarchy in (b) and (c).

matches P_2 between levels V_i and V_m, we see that between levels V_i and V_j, P must have had the cycle $u \rightsquigarrow v \rightsquigarrow e \rightsquigarrow b \rightsquigarrow u$, contradicting P being a tree pattern. By inspection, any other placement of $u \rightsquigarrow v$ to connect three of the four linking paths to form P_1 or P_2 similarly implies a cycle in P.

Hence, P cannot contain any more edges than those of P_α without matching P_1 or P_2. We observe that P_α consists of two paths sharing a common vertex x. Given the minimality of P in minimizing $|i - j|$, one path has both endpoints in V_i with one vertex in V_j that can be split into linking paths $L_1, L_2 \in \mathcal{L}(i, j)$. Similarly, the other has both endpoints in V_j with one vertex in V_i that can also be split into the linking paths $L_3, L_4 \in \mathcal{L}(i, j)$. In P_3 of Fig. 8(a), L_1 is $a \rightsquigarrow b$, L_2 is $b \rightsquigarrow e \rightsquigarrow x \rightsquigarrow c$, L_3 is $d \rightsquigarrow x \rightsquigarrow f \rightsquigarrow g$, and L_4 is $g \rightsquigarrow h$.

For P to be level nonplanar, a crossing must be forced between these two paths. This is done by having L_2 or L_3 meet the condition of P_3 of three subpaths $C_1 \in \mathcal{L}(i, m)$ linking V_i to V_m, $C_2 \in \mathcal{L}(l, m)$ linking V_l to V_m, and $C_3 \in \mathcal{L}(l, j)$ linking V_l to V_j. This is not the case for P_α in Fig. 8(a) since the $x \rightsquigarrow c$ portion of L_2 does not reach level V_m, and the $x \rightsquigarrow d$ portion of L_3 does not reach level V_l. So for P not to match P_3, at least one subpath of both L_2 and L_3 from x to V_i or V_j must strictly monotonic as was the case in Fig. 8(a). However, in this case P can be drawn without crossings. This leaves P_3 as the only possibility of a MLNP pattern matching P_α that does not match P_1 or P_2. □

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